Magnetic Monopoles, 't Hooft Lines, and the Geometric Langlands Correspondence

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Overview

Goal:

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To understand the Langlands correspondence in terms of topologically twisted $\mathcal{N} = 4$ super Yang-Mills gauge theory

Conjecture (Langlands)

To each n-dimensional representation of the absolute Galois group, there is a corresponding automorphic representation of $\mathrm{GL}_n(\mathbb{Q})$ *so that the Frobenius eigenvalues of the Galois representation agree with the Hecke eigenvalues of the automorphic representation.*

Q: What are Galois representations?

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A: They are n-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Q: What are automorphic representations?

Adeles

Definition (Ring of adeles)

The **ring of adeles** of $\mathbb{Q}$ is defined as

$$\mathbb{A}_{\mathbb{Q}} := \mathbb{R} \times \prod_{p \text{ prime}}^{res} \mathbb{Q}_p,$$

where $\mathbb{Q}_p$ denotes the p-adic completion of the rationals. Here $\mathbb{R}$ can be viewed as the completion at $p = \infty$ and the above product is *restricted* in the sense that:

$$\prod_{p \text{ prime}}^{res} \mathbb{Q}_p := \left\{ (x_p) \in \prod_{p \text{ prime}} \mathbb{Q}_p \mid x_p \in \mathbb{Z}_p \text{ for all but finitely many } p \right\}.$$

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Though not absolutely precise, this is a good first-order description of what an automorphic representation is.

Langlands over Finite Fields

Definition (Adele Ring for $\mathbb{F}_q(t)$)

The ring of adeles of $\mathbb{F}_q(t)$ is defined as

$$\mathbb{A}_{\mathbb{F}_q(t)} := \prod_{x \in \mathbb{P}^1(\mathbb{F}_q)}^{res} \mathbb{F}_q((t-x))$$

and the above product is restricted as before in the sense that all but finitely many terms in this product over x lie in $\mathbb{F}_q[[t-x]]$. Here the completion at the point at infinity corresponds to $\mathbb{F}_q((1/t))$.

Langlands over Finite Fields

We naturally have that

$$\mathbb{O}_{\mathbb{F}_q(t)} := \prod_{x \in \mathbb{P}^1(\mathbb{F}_q)} \mathbb{F}_q[[t - x]]$$

sits inside $\mathbb{A}_{\mathbb{F}_q(z)}$.

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Automorphic representations $\to \mathrm{GL}_n(\mathbb{O}_F)$-invariant functions on $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)$

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Automorphic representations $\rightarrow \mathrm{GL}_n(\mathbb{O}_F)$-invariant functions on $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)$

Galois representations $\rightarrow$ representations of **étale fundamental group** (in the unramified case)

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Theorem (Weil Uniformization)

Take F the function field for a curve C over $\mathbb{F}_q$. There is a canonical bijection as sets between

$$G(F)\backslash G(\mathbb{A}_F)/G(\mathbb{O}_F)$$

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Moreover, there exists an algebraic stack denoted by $\mathrm{Bun}_G(C)$ whose set of $\mathbb{F}_q$ points are in canonical bijective correspondence with this set.

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For C an unramified curve, a the étale fundamental group $\pi_1^{\text{ét}}$ corresponds to $\pi_1(C)$ with C over $\mathbb{C}$.

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$\rightarrow$ Flat connections on C

Meta-conjecture of Geometric Langlands

$$\mathcal{D}(\mathrm{Bun}_{\mathrm{G}}(C)) \cong \mathcal{QC}(\mathrm{Flat}_{\check{G}}(C)) \tag{1}$$

Q: How does this connect to physics?

Bosonic part of the action in $\mathcal{N} = 4$ super Yang-Mills

$$\frac{1}{e^2}\int_M \mathrm{Tr}\left(F \wedge \star F + \sum_i \mathrm{d}_A\phi \wedge \star(\mathrm{d}_A\phi) + \sum_{i<j}[\phi_i, \phi_j]^2 \mathrm{Vol}_M\right) \quad (2)$$

Concept (Montonen-Olive Duality)

In 4D $\mathcal{N}=4$ supersymmetric Yang-Mills theory with gauge group G and complex coupling constant τ, any correlator of observables

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle_{\tau,G} := \int \mathcal{D}\{Fields\}\, \mathcal{O}_1 \dots \mathcal{O}_n\, e^{-S}$$

can be rewritten in terms of Yang-Mills theory with inverse coupling constant $-1/n_{\mathfrak{g}}\tau$ on the Langlands dual group $\check{G}$ as a correlator of dual operators $\tilde{\mathcal{O}}_1 \dots \tilde{\mathcal{O}}_n$

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle_{\tau,G} = \left\langle \tilde{\mathcal{O}}_1 \dots \tilde{\mathcal{O}}_n \right\rangle_{-1/n_{\mathfrak{g}}\tau,\check{G}}.$$

Topological Twisting

Physical Concept (Topological Twist)

Given a supersymmetric (SUSY) field theory $\mathcal{E}$, a topological twist is a procedure for extracting a sector of $\mathcal{E}$ that depends only on the topology of the spacetime manifold. The resulting field theory is **topological**.

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In the topological twist, the action becomes:

$$S = \{Q, V\} + \frac{i\theta}{8\pi^2} \int_M \mathrm{Tr}\,(F \wedge F) - \frac{1}{e^2} \frac{v^2 - u^2}{v^2 + u^2} \int_M \mathrm{Tr}\,(F \wedge F). \quad (3)$$

$\Psi := \frac{\theta}{2\pi} + \frac{v^2-u^2}{v^2+u^2}\frac{4\pi i}{e^2}$ is the **Kapustin-Witten parameter**.

Equations of Motion in the Twisted 4D Theory

$$
\begin{aligned}
(F - \phi \wedge \phi + tD_\phi)^+ &= 0 \\
(F - \phi \wedge \phi - t^{-1} D_\phi)^- &= 0 \\
D \star \phi &= 0 \\
\sigma &= 0
\end{aligned}
\tag{4}
$$

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$$F - \phi \wedge \phi + \star D\phi = 0, \qquad D \star \phi = 0. \tag{5}$$

"Bogomolny like"

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$$\mathcal{F} = 0.$$

Definition (Wilson Loop)

Given a field theory with gauge group G and a finite-dimensional representation R of G together with a closed loop γ, we define the Wilson loop operator:

$$\mathcal{W}_R(\gamma) := \mathrm{Tr}\, R(\mathrm{Hol}(A, \gamma)). \tag{7}$$

Operator-Product Expansion of Wilson Lines

Because of supersymmetry, the limit $\lim_{\gamma \to \gamma'} W_R(\gamma) W_{R'}(\gamma')$ can be evaluated classically.

$$\lim_{\gamma \to \gamma'} W_R(\gamma) W_{R'}(\gamma') = \sum_{\substack{\alpha \\ \text{irrep.}}} n_\alpha W_{R_\alpha}(L').$$

This will act as *Satake symmetries* on the Galois side.

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$$S[A] := \int_M \mathrm{Tr}\left(F \wedge \star F\right) \tag{8}$$

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- Instantons must satisfy the (anti)-self duality equations

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- The space of instanton solutions of *finite action* was constructed by Atiyah, Hitchin, Drinfeld, and Mannin: the ADHM construction

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- These are the **Bogomolny equations** for magnetic monopoles
- Again have an invariant **monopole number** for a solution to these equations.

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- Let S_R^2 be a two-sphere of radius R in $\mathbb{R}^3$.
- The insertion of monopoles inside S_R^2 will modify the G-bundle over S_R^2 to have nontrivial Chern classes

Fact

Representations of $\check{G}$ classify the G-bundles on $\mathbb{CP}^1 = S^2$.

From 't Hooft lines to Monopoles

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- Take $M=\mathbb{R}\times I\times C$, with $\mathbb{R}$ the "time" direction and take a Hamiltonian point of view on $W=I\times C$.
- We can locally take $\phi=\phi_4 dx^4$ so that on W, ϕ behaves as a scalar.
- Then, on W, the A-model equations reduce exactly to the Bogomolny equations for monopoles:

$$F=\star_3 D_A\phi.$$

- Write a local coordinate $z \in \mathbb{C}$ parameterizing C and $\sigma \in \mathbb{R}$ parameterizing I. We can gauge away $A_\sigma = 0$.

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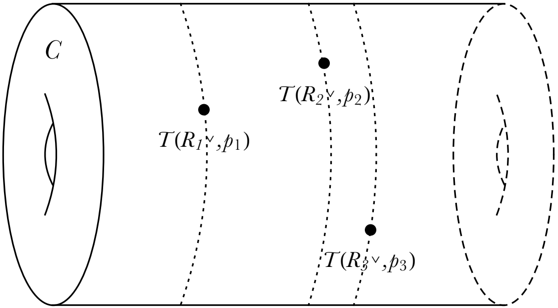
- Can be interpretted as saying that the holomorphic class of the G-bundle over C remains constant *away from singularities*.


C
$T(R_1{}^\vee, p_1)$
$T(R_2{}^\vee, p_2)$
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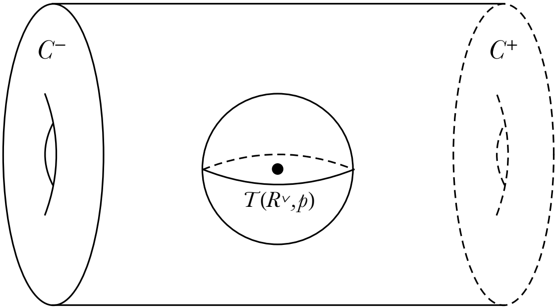
- We denote this space by $\mathcal{Z}(\check{R}_1, p_1, \dots, \check{R}_k, p_k)$.
- On general grounds we can show that it is independent of the p_i and factors into a product:

$$\mathcal{Z}(\check{R}_1, \dots, \check{R}_k) = \prod_i \mathcal{Z}(\check{R}_i).$$


C^-
C^+
$T(R^\vee, p)$

The solution space of the Bogomolny equations for a 't Hooft insertion of type $\check{R}_i$ is equivalent to the Schubert cell corresponding to $\check{R}_i$ in the affine Grassmannian:

$$\mathcal{Z}(\check{R}_i) \cong \mathcal{N}(\check{R}_i) \subset Gr_G.$$

Our "Hilbert Space" of states will be obtained from taking (intersection) cohomology of the space of solutions to the Bogomolny equations, i.e.

$$\mathcal{H}(\check{R}_1, \dots \check{R}_k) := H^\bullet(\overline{\mathcal{Z}}(\check{R}_1, \dots, \check{R}_k)) \tag{10}$$

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This gives the relationship

$$\check{R} \leftrightarrow H^\bullet(\mathcal{N}(\check{R})).$$