Instantons and the ADHM Construction

Prerequisite Material: Fiber Bundles

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October 25, 2016

Abstract

1 Fiber Bundles

1.1 Definitions and Examples

We are working on a manifold $M$ which we will call our base space. On this, we have a coordinate bundle:

**Definition 1 (Coordinate Bundle).** A coordinate bundle consists of

- A total space $E$
- A base space $M$
- A fiber $F$
- A surjection $\pi : E \to M$ called projection to a point $p$ on $M$ so that $\pi^{-1}(p) := E_p \cong F$. This is the fiber over $p$.
- A Lie Group $G$ freely acting on the fiber: $G \circ F$ s.t. $gf = f \Rightarrow g = 1 \forall f \in F$.
- A set of open coverings $\{U_\alpha\}_{\alpha \in I}$ of $M$ with diffeomorphisms $\phi_\alpha : U_\alpha \times F \to$
\[ \pi^{-1}(U_i) \text{ called local trivializations} \] so that the following diagram commutes.

\[ \begin{array}{ccc}
U_\alpha \times F & \xrightarrow{\psi_\alpha} & \pi^{-1}(U_\alpha) \\
\downarrow{p_1} & & \downarrow{\pi} \\
U_\alpha & & 
\end{array} \]

- On \( p \in U_{\alpha\beta} := U_\alpha \cap U_\beta \), \( \psi_\beta^{-1}\psi_\alpha \) acts as a diffeomorphism coinciding with the action of an element of \( G \) on each \( E_p \) (we say “fiberwise”).

In this way \( \psi_\alpha \) gives rise to a diffeomorphism between \( F \) and \( F_p \) given by \( \psi_{\alpha,F_p}(f) = \psi_\alpha(p,x) \)

Fiber bundles generalize the product of two spaces by allowing for local product structure but much more interesting global “twisted structure”.

From this we can define the vertical component of a point in the total space: \( f_\alpha : E \rightarrow F \) by \( f_\alpha = \psi_{\alpha^{-1}} \).

We can also identify \( \psi_{\beta,p}^{-1} \circ \psi_{\alpha,p} \) with an element in \( G \) by \( g_{\alpha,\beta} : U_{\alpha\beta} \rightarrow G \).

**Proposition 2.** \( g_{\alpha\beta} \) satisfies

- \( g_{\alpha\alpha} = 1 \)
- \( g_{\alpha\beta} = g_{\beta\alpha}^{-1} \)
- \( g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma} \)

Moreover

1. \( g_{\alpha\beta}f_\beta = f_\alpha \)
   That is, \( g_{\alpha\beta} \) maps the fiber corresponding to \( U_\beta \) to the fiber corresponding to \( U_\alpha \).

2. \( \psi_j(p,f) = \psi_i(p,g_{ij}f) \)

**Proof.** These are all easy to check just by the definition of \( g_{\alpha\beta} \) as a composition of the \( \psi_\alpha \) and by invoking the cartesian properties of local trivialization.

The equivalence class of a coordinate bundle on \( M \) is called a **fiber bundle** over \( M \).

Fiber bundles whose fibers are vector spaces are called vector bundles. Examples are the tangent/cotangent spaces to a manifold, and any tensor/symmetric/exterior powers thereof. We will see that we can view vector fields, \( p \)-forms, and many other interesting, physically-relevant, objects as “slices” or **sections** of fiber bundles. We will advance this idea shortly.
1.2 Principal Bundles

When the fiber $F$ is the structure group itself: $F = G$, then $G$ obviously has standard left action on $F$, and we get the principal bundle $P(M, G)$. That is, over every point is fibered a copy of $G$. This will be an object of central interest in the following lectures.

**Proposition 3.** The principal bundle is equipped with a natural right action of $G$, $R_g$ so that $R_g : \pi^{-1}(U_\alpha) \rightarrow \pi^{-1}(U_\alpha)$ by acting on the fiber appropriately $R_g(p, h) = (p, hg)$. It acts smoothly and freely on the principal bundle.

We state the following theorem without proof (c.f. Chapter 9 of Lee’s Introduction to Smooth Manifolds)

**Theorem 4.** When $G$ is a compact Lie Group acting smoothly and freely on a manifold $M$, the orbit space $M/G$ is a topological manifold with dimension $\dim M - \dim G$ and a unique smooth structure so that $\pi : M \rightarrow M/G$ is a smooth submersion (differential is locally surjective).

Otherwise we get “orbitfolds” (consider $\mathbb{H}/\text{PSL}_2(\mathbb{R})$)

**Corollary 5.** For the principal bundle $P(M, G)$ we get $\dim P = \dim M + \dim G$

If $M, F$ are two manifolds and $G$ has an action $G \times F \rightarrow F$, then for an open cover $\{U_\alpha\}$ of $M$ with a map $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow G$ we can construct a fiber bundle by first building the set

$$X = \bigcup_\alpha U_\alpha \times F$$

(1)

and quotienting out by the relation

$$(x, f) \in U_\alpha \times F \sim (x', f') \in U_\beta \times F \iff x = x', f = g_{\alpha\beta}(x)f'$$

(2)

Then $E = X/\sim$ is a fiber bundle over $M$. We can denote elements of $E$ by $[x, f]$ so that

$$\pi(x, f) = x, \quad \psi_\alpha(x, f) = [x, f].$$

(3)

**Proposition 6.** For a fiber bundle $\pi : E \rightarrow M$ with overlap functions $g_{\alpha\beta} : U_{\alpha\beta} \rightarrow G$ between charts, we can form a principal bundle $P(M, G)$ so that

$$P = X/\sim, \quad X = \bigcup_\alpha U_\alpha \times G$$

(4)

Note that there was no requirement here that $G$ be compact. We will often deal with $G$ compact in the future lectures, but when looking at hyperbolic Riemann surfaces, it not the case that $G$ is usually compact.
1.3 Morphisms and Extensions

The morphisms in the category of fiber bundles are called bundle maps:

**Definition 7 (Bundle Map).** For two fiber bundles $\pi : E \to M, \pi' : E' \to M'$ a bundle map is a smooth map $\bar{f} : E \to E'$ that naturally induces a smooth map on the base spaces so that the following diagram commutes:

\[
\begin{array}{ccc}
E & \xrightarrow{\bar{f}} & E' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
M & \xrightarrow{f} & M
\end{array}
\]

Two bundles are equivalent if there is a bundle map so that both $\bar{f}$ and $f$ are diffeomorphisms.

If we have a fiber bundle $\pi : E \to M$ and $\phi : N \to M$ for another manifold $N$, then we can pull back $E$ to form a bundle over $N$.

\[\phi^*E = \{(y, [f, p]) \in N \times E s.t. \phi(y) = p\}\] (5)

We have projection on the second factor of $\phi^*E$ as a map $g : \phi^*E \to E$.

This is the pullback bundle $\phi^*E$.

**Definition 8 (Pullback Bundle).** For a map $\phi : N \to M$ and $E$ a fiber bundle over $M$ so that $\pi : E \to M$, we define the pullback bundle $\phi^*M$ so that the following diagram commutes:

\[
\begin{array}{ccc}
\phi^*E & \xrightarrow{g} & E \\
\downarrow{\pi'} & & \downarrow{\pi} \\
N & \xrightarrow{\phi} & M
\end{array}
\]

We can take products of these bundles as topological spaces in the obvious way:

\[E \times E' \xrightarrow{\pi \times \pi'} M \times M'\] (6)

In the special case where $M = M'$ we get

**Definition 9 (Direct Sum of Vector Bundles).** For $E, E'$ vector bundles over $M$ we can define their sum as $E \oplus E'$ to be $M$ with $F \oplus F'$ fibred over every point.

More compactly, it is the pullback bundle of the map $f : M \to M \times M$

The structure group of $E \oplus E'$ is the product $G \times G'$ of the structure groups of the original bundles and it acts diagonally on their sum.

\[G^{E \oplus E'} = \left\{ \begin{pmatrix} g^E & 0 \\ 0 & g^{E'} \end{pmatrix} : g^E \in G, g^{E'} \in G' \right\}\] (7)

and the transition functions act diagonally in the same way.
Alternatively we could have defined
\[ \bar{E} = \{(p, f), [p, f'] \in E \times E'\} \] (8)
which is a bundle over \(M\) as well, and its easy to see this is also the direct sum bundle.
Similarly, we can define arbitrary direct sums of bundles recursively:
\[ E_1 \oplus \cdots \oplus E_r \] (9)

For some intuition about when fiber bundles are nontrivial, consider the following theorem

**Theorem 10.** Let \( \pi : E \to M \) be a fiber bundle over \( M \) and consider maps \( f, g \) from \( N \to M \) so that \( f, g \) are homotopic, then the pullback bundles are equivalent: \( f^* E \cong g^* E \) over \( N \).

**Corollary 11.** If \( M \) is contractible, every fiber bundle \( \pi : E \to M \) is trivial.

### 1.4 Sections and Lifts

As mentioned before, any specific smooth vector field on a manifold \( M \) can be viewed as a smooth “slice” of the vector bundle of the tangent spaces of \( M: TM \). This motivates the notion of a section of a fiber bundle that associates to each base point \( p \in M \) an element \( f \) in the fiber \( F_p \), giving together \( (p, f) \in E \).

A **global section** of the fiber bundle \( \pi : E \to M \) is a map \( s : M \to E \) so that \( \pi \circ s = \text{id} \). When it’s restricted, \( s : U \subseteq M \to E \), we call \( s \) a **local section**. The set of smooth global sections is denoted by \( \Gamma^\infty(M, E) \).

**Example 12.** The set of all smooth \( r \)-forms on \( M \) is \( \Gamma^\infty(M, \Lambda^r(T^*M)) \) on which the structure group acts on each wedge. Note the different action of the structure group on different \( r \)-forms is exactly what makes the components of various \( r \)-forms “\( r \)-times covariant”.

When the group is fibered over the manifold, then on the local cartesian structure, we can easily pick the section \( p \mapsto [p, e] \).

**Proposition 13.** For a principal bundle \( P(M, G) \), any local trivialization \( \psi : U \times G \to \pi^{-1}(U) \) defines a local section by \( s : p \mapsto \psi(p, e) \) and conversely any local section defines a trivialization by \( \psi(p, g) = s(p)g \)

By using sections, we can prove the existence of lifts. That is, for a principal bundle \( P(M, G) \) over \( M \), and a map \( \varphi : M \to N \) we can get a principal bundle over \( N \) by forming the projection \( \varphi \circ \pi \).
**Proposition 14.** For a principal bundle $P(M,G)$ and $\varphi : M \to N$, then $\varphi$ is smooth iff $\varphi \circ \pi$ is smooth according to the following diagram.

\[
P(M,G) \xrightarrow{\pi} M \xrightarrow{\varphi} N \xrightarrow{\varphi \circ \pi} N
\]

*Proof.* If $\varphi$ is smooth, then $\varphi \circ \pi$ is a composition of smooth maps. On the other hand, if $\varphi \circ \pi$ is smooth, then for each point $p$ there is a coordinate neighborhood $U_\alpha$ on which we have trivial fiber structure. Take a local section $s_\alpha$ so that $\varphi \circ \pi \circ s = \varphi|_{U_\alpha}$.

**Proposition 15.** For $P(M,G)$ principal and $\tilde{\varphi} : P(M,G) \to N$ a smooth $G$-invariant map so that

\[
\tilde{\varphi}(xg) = \tilde{\varphi}(x), \quad x \in P(M,G)
\]

then there is a unique map $\tilde{\varphi}$ induced on the base space so that the following diagram commutes:

\[
P(M,G) \xrightarrow{\pi} M \xrightarrow{\tilde{\varphi}} N
\]

and is given by $\tilde{\varphi}([x,g]) = \varphi(x)$. This is well-defined.

## 2 Lie Groups and Algebras

Although standard knowledge on the definition of a Lie Group/Algebra is assumed, let’s try to motivate the ideas within this field in a more geometric way than is often done.

Consider a manifold $M$, and consider $\text{Vect}(M)$, the space of all smooth vector fields on $M$. For a map $\varphi : M \to N$ we have a notion of **pushforward** $\varphi_* : \text{Vect}(M) \to \text{Vect}(N)$ on vector fields given by their actions on functions as

\[
[\varphi_*(v)](f) = v(\varphi^*(f))
\]

A smooth vector field $X$ on $M$ gives rise to **flows** that are solutions to the differential equation of motion

\[
\frac{d}{dt}f(\gamma(t)) = Xf.
\]

One could argue, more strongly, that in fact the entire field of ordinary differential equations has an interpretation as equations of motion along flows of vector fields. Such a viewpoint has brought forward the lucrative insights of symplectic geometry.
The motion along this flow is expressed as the exponential:

\[ f(\gamma(t)) = e^{tX}f(p), \quad p = \gamma(0) \]  

(13)

Now consider two vector fields \( X, Y \) on \( M \). Let \( Y \) flow along \( X \) so we move along \( X \) giving:

\[ e^{tX}Y = Y(\gamma(t)) \in T_{\gamma(t)}M \]  

(14)

Note that the reverse flow \( e^{-tX} \) maps \( T_{\gamma(t)}M \to T_{\gamma(0)}M = T_pM \), so acts by pushforward on \( e^{tX}Y \) equivalent to:

\[ e^{tX}Ye^{-tX} \in T_p \]  

(15)

We can compare this to \( Y \) and take the local change by dividing through by \( t \) as \( t \to 0 \), giving the Lie derivative

\[ \mathcal{L}_X Y := \frac{e^{tX}Ye^{-tX} - Y}{t} \]  

(16)

It is easy to check that this is in fact antisymmetric and gives rise to a bilinear form on \( \text{Vect}(M) \)

\[ [X,Y] := \mathcal{L}_X Y \]  

(17)

A vector space endowed with such a bilinear form and satisfying the Jacobi identity is a Lie algebra.

Most important is when \( M \) itself has group structure, so is a Lie group, which we will denote by \( G \). Then the vector fields on \( G \) of course also form a Lie algebra, just by virtue of the manifold structure of \( G \).

We state the following proposition without proof

**Proposition 16.** Let \( \varphi : M \to N \) be a diffeomorphism of Lie groups. Then \( \varphi_* : \text{Vect}(M) \to \text{Vect}(N) \) is a homomorphism of Lie algebras.

For a Lie group, group elements induce automorphisms on the manifold by left multiplication, denoted \( L_g \) and by right multiplication \( R_g \):

\[ R_g : G \to G, \quad g : h \mapsto gh \]

\[ L_g : G \to G, \quad g : h \mapsto hg \]  

(18)

We have that each group element induces (by pushforward) a map between tangent spaces

\[ (L_g)_* : T_hG \to T_{gh}G \]

\[ (R_g)_* : T_hG \to T_{hg}G \]  

(19)

A vector field \( X \) is left-invariant if \( (L_g)_*X(h) = X(gh) \).

By the proposition, we get that \( (L_g)_*[X,Y] = [(L_g)_*X,(L_g)_*Y] \) so these left-invariant vector fields in fact form a Lie algebra for the group. It exactly the vector fields representing the symmetries of \( G \).
In local coordinates, the commutator can be written as:

\[ X = X^\mu \partial _\mu , \quad Y = Y^\mu \partial _\mu \]

\[ [X, Y] = (X^\nu \partial _\nu Y^\mu - Y^\nu \partial _\nu X^\mu ) \partial _\mu \] \hspace{1cm} (20)

Left-invariant vectors flow consistent with the group action:

\[ (L_g)_* X(e) = X(g) \] \hspace{1cm} (21)

The set of all left-invariant vector fields can be uniquely extracted from their value at the identity by this rule, and in fact for any vector \( x \in T_eG \), there is a corresponding left-invariant vector field \( X(g) = (L_g)_* x \). Therefore the tangent space to the identity gives rise to a Lie algebra which we will call the Lie algebra of \( G \) and denote by \( \mathfrak{g} \). This Lie algebra (often referred to as the Lie algebra \( \mathfrak{g} \) associated to the group \( G \)) is finite dimensional when \( G \) is.

Now because we define the Lie algebra as the “tangent space to the identity”, it is worth asking “how does the Lie algebra appear at a generic point, \( g \), on the group?”. The idea is to bring that vector back to the identity using \( G \) and see what it looks like.

This is accomplished by using the **Maurer-Cartan form** \( \Theta \), which is a \( \mathfrak{g} \)-valued 1-form on \( G \) so that

\[ \Theta (g) = (L_{g^{-1}})_* \] \hspace{1cm} (22)

Note that this maps from \( \text{Vect}(G) \to \mathfrak{g} \). It takes a vector \( v \) at point \( g \) and traces it back to the natural vector at the identity that would have gotten pushed forward to \( v \) under \( g \).

**Proposition 17** (Properties of exp). For \( G \) a compact and connected Lie group, with Lie algebra \( \mathfrak{g} \), we have a map \( \text{exp} : \mathfrak{g} \to G \).

1. \([X, Y] = 0 \iff e^X e^Y = e^Y e^X\)

2. The map \( t \to \text{exp}(tX) \) is a homomorphism from \( \mathbb{R} \) to \( G \).

3. If \( G \) is connected then \( \text{exp} \) generates \( G \) as a group, meaning all elements can be written as some product \( \exp(X_1) \ldots \exp(X_n) \) for \( X_i \in \mathfrak{g} \)

4. If \( G \) is connected and compact then \( \text{exp} \) is surjective. It is almost never injective.

**Example 18.** The Lie algebra associated to the Lie group \( \text{U}(n) \) of unitary matrices is \( \text{u}(n) \) of antihermitian matrices. This is the same as the Lie algebra for the group \( \text{SU}(n) \)

**Definition 19** (Adjoint Action on \( G \)). For each \( g \) we can consider the homomorphism \( \text{Ad}_g : h \mapsto ghg^{-1} \) or \( \text{Ad}_g = L_g \circ R_{g^{-1}} \). This defines a representation

\[ \text{Ad} : g \to \text{Diff}(G) \] \hspace{1cm} (23)
Definition 20 (Adjoint Representation of $\mathfrak{g}$). The pushforward of this action gives rise to the adjoint representation of the Lie group $\mathfrak{g}$ by

$$ (\text{Ad}_g)_* = (L_g \circ R_{g^{-1}})_* $$

From the product rule, this acts as $[g, -]$ at the identity. We denote this as

$$ \text{ad} : \mathfrak{g} \to \text{End} \mathfrak{g} $$

The Jacobi identity ensures that ad is a homomorphism. If the center of $G$ is zero then ad is faithful and we have an embedding into $\text{GL}(n)$. This is nice because it also shows that modulo a central extension, every Lie algebra can be represented into $\text{GL}(n)$, a weaker form of Ado’s theorem.

Moreover the adjoint representation gives rise to a natural metric on $G$ called the Killing Form given by

$$ \kappa(X,Y) = \text{Tr}(\text{ad}(X)\text{ad}(Y)) $$

3 Associated Bundles

Take a principal bundle $P(M,G)$ and let $F$ be a space with associated automorphism $\text{Aut}(F)$ so that $\rho : G \to \text{Aut}(F)$ is a faithful representation. Then $g \cdot f$ is a well-defined notion, with free action, and we can consider the (right) action of $G$ on $P(M,G) \times F$ by

$$ g \cdot ([p,h],f) = ([p,hg], \rho(g)^{-1}f) $$

This is a free action as well. Then if $G$ is compact (important) we have the orbit space

$$ E_\rho = P(M,G) \times F/G $$

is a manifold.

Theorem 21. The space $E_\rho$ can be made into a fiber bundle over $M$ with fiber $F$ called the associated fiber bundle of $P(M,G)$.

Proof. We make $P \times F$ into a bundle by defining the projection

$$ \pi_\rho([p,h],f) = p $$

and trivializations $\psi_\alpha : U_\alpha \times F \to \pi^{-1}(U)\alpha$ by

$$ (\psi_\rho)_\alpha(p,f) = ([p,s_\alpha(p)],f) $$

and inverse

$$ (\psi_\rho)_\alpha^{-1}([p,g],f) = [p,\rho(g)f] $$

From this, if $F$ is a group then we can make $\pi_\rho^{-1}(p)$ into a group at each fiber in the obvious way, defining $[(p,v)][p,w] = [p,vw]$. And if $F$ is a vector space then we can do the same construction to make each fiber have vector space structure.

The two associated bundles that we’ll care about are $P(M,G) \times_{\text{Ad}} G$ and $P(M,G) \times_{\text{ad}} \mathfrak{g}$. 