Instantons and the ADHM Construction

Prerequisite Material: Fiber Bundles

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Abstract

1 Fiber Bundles

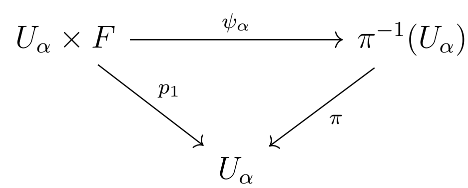
1.1 Definitions and Examples

We are working on a manifold M which we will call our **base space**. On this, we have a **coordinate bundle**:

Definition 1 (Coordinate Bundle)**.** A coordinate bundle consists of

- A **total space** E
- A base space M
- A **fiber** F
- A surjection $\pi : E \to M$ called **projection** to a point p on M so that $\pi^{-1}(p) := E_p \cong F$. This is the fiber over p.
- A **Lie Group** G freely acting on the fiber: $G \circlearrowleft F$ s.t. $gf = f \Rightarrow g = 1 \forall f \in F$.
- A set of open coverings $\{U_\alpha\}_{\alpha \in I}$ of M with diffeomorphisms $\phi_\alpha : U_\alpha \times F \to$

$\pi^{-1}(U_i)$ called **local trivializations** so that the following diagram commutes.



- On $p \in U_{\alpha\beta} := U_\alpha \cap U_\beta$, $\psi_\beta^{-1}\psi_\alpha$ acts as a diffeomorphism coinciding with the action of an element of G on each E_p (we say "fiberwise").

In this way ψ_α gives rise to a diffeomorphism between F and F_p given by $\psi_{\alpha,F_p}(f) = \psi_\alpha(p,x)$

Fiber bundles generalize the product of two spaces by allowing for local product structure but much more interesting global "twisted structure".

From this we can define the vertical component of a point in the total space: $f_\alpha : E \to F$ by $f_\alpha = \psi_{\alpha,\pi}^{-1}$.

We can also identify $\psi_{\beta,p}^{-1} \circ \psi_{\alpha,p}$ with an element in G by $g_{\alpha,\beta} : U_{\alpha\beta} \to G$.

Proposition 2. *$g_{\alpha\beta}$ satisfies*

- $g_{\alpha\alpha} = 1$
- $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$
- $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$

Moreover

1. $g_{\alpha\beta}f_\beta = f_\alpha$
   That is, $g_{\alpha\beta}$ maps the fiber corresponding to U_β to the fiber corresponding to U_α.
2. $\psi_j(p,f) = \psi_i(p, g_{ij}f)$

Proof. These are all easy to check just by the definition of $g_{\alpha\beta}$ as a composition of the ψ_α and by invoking the cartesian properties of local trivialization. □

The equivalence class of a coordinate bundle on M is called a **fiber bundle** over M.

Fiber bundles whose fibers are are vector spaces are called vector bundles. Examples are the tangent/cotangent spaces to a manifold, and any tensor/symmetric/exterior powers thereof. We will see that we can view vector fields, p-forms, and many other interesting, physically-relevant, objects as "slices" or **sections** of fiber bundles. We will advance this idea shortly.

1.2 Principal Bundles

When the fiber F is the structure group itself: $F = G$, then G obviously has standard left action on F, and we get the **principal bundle** $P(M, G)$. That is, over every point is fibered a copy of G. This will be an object of central interest in the following lectures.

Proposition 3. *The principal bundle is equipped with a natural right action of G, R_g so that $R_g : \pi^{-1}(U_\alpha) \to \pi^{-1}(U_\alpha)$ by acting on the fiber appropriately $R_g(p, h) = (p, hg)$.* It acts smoothly and freely on the principal bundle.

We state the following theorem without proof (c.f. Chapter 9 of Lee's Introduction to Smooth Manifolds)

Theorem 4. *When G is a compact Lie Group acting smoothly and freely on a manifold M, the orbit space M/G is a topological manifold with dimension $\dim M - \dim G$ and a unique smooth structure so that $\pi : M \to M/G$ is a smooth submersion (differential is locally surjective).*

Otherwise we get "orbitfolds" (consider $\mathbb{H}/\mathrm{PSL}_2(\mathbb{R})$)

Corollary 5. *For the principal bundle $P(M, G)$ we get $\dim P = \dim M + \dim G$*

If M, F are two manifolds and G has an action $G \times F \to F$, then for an open cover $\{U_\alpha\}$ of M with a map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ we can construct a fiber bundle by first building the set

$$X = \bigcup_\alpha U_\alpha \times F \tag{1}$$

and quotienting out by the relation

$$(x, f) \in U_\alpha \times F \sim (x', f') \in U_\beta \times F \iff x = x', f = g_{\alpha\beta}(x) f' \tag{2}$$

Then $E = X/\sim$ is a fiber bundle over M. We can denote elements of E by $[x, f]$ so that

$$\pi(x, f) = x,\ \psi_\alpha(x, f) = [x, f]. \tag{3}$$

Proposition 6. *For a fiber bundle $\pi : E \to M$ with overlap functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$ between charts, we can form a principal bundle $P(M, G)$ so that*

$$P = X/\sim,\ X = \bigcup_\alpha U_\alpha \times G \tag{4}$$

Note that there was no requirement here that G be compact. We will often deal with G compact in the future lectures, but when looking at hyperbolic Riemann surfaces, it not the case that G is usually compact.

1.3 Morphisms and Extensions

The morphisms in the category of fiber bundles are called **bundle maps**:

Definition 7 (Bundle Map)**.** For two fiber bundles $\pi : E \to M, \pi' : E' \to M'$ a bundle map is a smooth map $\bar{f} : E \to E'$ that naturally induces a smooth map on the base spaces so that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\bar{f}} & E' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{f} & M \end{array}$$

Two bundles are equivalent if there is a bundle map so that both $\bar{f}$ and f are diffeomorphisms.

If we have a fiber bundle $\pi : E \to M$ and $\varphi : N \to M$ for another manifold N, then we can pull back E to form a bundle over N.

$$\varphi^* E = \{(y, [f, p]) \in N \times E s.t. \varphi(y) = p\} \tag{5}$$

We have projection on the second factor of $\varphi^* E$ as a map $g : \varphi^* E \to E$.

This is the **pullback bundle** $\varphi^* E$.

Definition 8 (Pullback Bundle)**.** For a map $\varphi : N \to M$ and E a fiber bundle over M so that $\pi : E \to M$, we define the pullback bundle $\varphi^* M$ so that the following diagram commutes:

$$\begin{array}{ccc} \varphi^* E & \xrightarrow{g} & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{\varphi} & M \end{array}$$

We can take products of these bundles as topological spaces in the obvious way:

$$E \times E' \xrightarrow{\pi \times \pi'} M \times M' \tag{6}$$

In the special case where $M = M'$ we get

Definition 9 (Direct Sum of Vector Bundles)**.** For E, E' vector bundles over M we can define their sum as $E \oplus E'$ to be M with $F \oplus F'$ fibred over every point.

More compactly, it is the pullback bundle of the map $f : M \to M \times M$

The structure group of $E \oplus E'$ is the product $G \times G'$ of the structure groups of the original bundles and it acts diagonally on their sum.

$$G^{E \oplus E'} = \left\{ \begin{pmatrix} g^E & 0 \\ 0 & g^{E'} \end{pmatrix} : g^E \in G, g^{E'} \in G' \right\} \tag{7}$$

and the transition functions act diagonally in the same way.

Alternatively we could have defined

$$\bar{E} = \{([p, f], [p, f']) \in E \times E'\} \tag{8}$$

which is a bundle over M as well, and its easy to see this is also the direct sum bundle.

Similarly, we can define arbitrary direct sums of bundles recursively:

$$E_1 \oplus \cdots \oplus E_r \tag{9}$$

For some intuition about when fiber bundles are *nontrivial*, consider the following theorem

Theorem 10. *Let $\pi : E \to M$ be a fiber bundle over M and consider maps f, g from $N \to M$ so that f, g are homotopic, then the pullback bundles are equivalent: $f^*E \cong g^*E$ over N.*

Corollary 11. *If M is contractible, every fiber bundle $\pi : E \to M$ is trivial.*

1.4 Sections and Lifts

As mentioned before, any specific smooth vector field on a manifold M can be viewed as a smooth "slice" of the vector bundle of the tangent spaces of M: TM. This motivates the notion of a **section** of a fiber bundle that associates to each base point $p \in M$ an element f in the fiber F_p, giving together $(p, f) \in E$.

A **global section** of the fiber bundle $\pi : E \to M$ is a map $s : M \to E$ so that $\pi \circ s = \text{id}$. When it's restricted, $s : U \subseteq M \to E$, we call s a **local section**. The set of smooth global sections is denoted by $\Gamma^\infty(M, E)$.

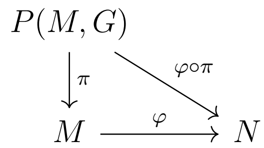
Example 12. The set of all smooth r-forms on M is $\Gamma^\infty(M, \Lambda^r(T^*M))$ on which the structure group acts on each wedge. Note the different action of the structure group on different r-forms is exactly what makes the components of various r-forms "r-times covariant".

When the group is fibered over the manifold, then on the local cartesian structure, we can easily pick the section $p \mapsto [p, e]$.

Proposition 13. *For a principal bundle $P(M, G)$, any local trivialization $\psi : U \times G \to \pi^{-1}(U)$ defines a local section by $s : p \mapsto \psi(p, e)$ and conversely any local section defines a trivialization by $\psi(p, g) = s(p)g$*

By using sections, we can prove the existence of lifts. That is, for a principal bundle $P(M, G)$ over M, and a map $\varphi : M \to N$ we can get a principal bundle over N by forming the projection $\varphi \circ \pi$.

Proposition 14. *For a principal bundle $P(M,G)$ and $\varphi : M \to N$, then φ is smooth iff $\varphi \circ \pi$ is smooth according to the following diagram.*

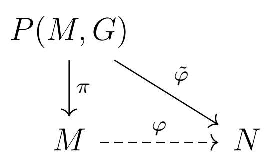


Proof. If φ is smooth, then $\varphi \circ \pi$ is a composition of smooth maps. On the other hand, if $\varphi \circ \pi$ is smooth, then for each point p there is a coordinate neighborhood U_α on which we have trivial fiber structure. Take a local section s_α so that $\varphi \circ \pi \circ s = \varphi|_{U_\alpha}$. □

Proposition 15. *For $P(M,G)$ principal and $\tilde{\varphi} : P(M,G) \to N$ a smooth G-invariant map so that*

$$\tilde{\varphi}(xg) = \tilde{\phi}(x), \;\; x \in P(M,G) \tag{10}$$

then there is a unique map ϕ induced on the base space so that the following diagram commutes:



and is given by $\tilde{\phi}([x,g]) = \varphi(x)$. This is well-defined.

2 Lie Groups and Algebras

Although standard knowledge on the definition of a Lie Group/Algebra is assumed, let's try to motivate the ideas within this field in a more geometric way than is often done.

Consider a manifold M, and consider $\mathrm{Vect}(M)$, the space of all smooth vector fields on M. For a map $\varphi : M \to N$ we have a notion of **pushforward** $\varphi_* : \mathrm{Vect}(M) \to \mathrm{Vect}(N)$ on vector fields given by their actions on functions as

$$[\varphi_*(v)](f) = v(\varphi^*(f)) \tag{11}$$

A smooth vector field X on M gives rise to **flows** that are solutions to the differential equation of motion

$$\frac{d}{dt} f(\gamma(t)) = Xf. \tag{12}$$

One could argue, more strongly, that in fact the *entire field* of ordinary differential equations has an interpretation as equations of motion along flows of vector fields. Such a viewpoint has brought forward the lucrative insights of symplectic geometry.

The motion along this flow is expressed as the exponential:

$$f(\gamma(t)) = e^{tX} f(p),\ p = \gamma(0) \tag{13}$$

Now consider two vector fields X, Y on M. Let Y flow along X so we move along X giving:

$$e^{tX} Y = Y(\gamma(t)) \in T_{\gamma(t)} M \tag{14}$$

Note that the reverse flow e^{-tX} maps $T_{\gamma(t)}M \to T_{\gamma(0)}M = T_p M$, so acts by pushforward on $e^{tX}Y$ equivalent to:

$$e^{tX} Y e^{-tX} \in T_p \tag{15}$$

We can compare this to Y and take the local change by dividing through by t as $t \to 0$, giving the Lie derivative

$$\mathcal{L}_X Y := \frac{e^{tX} Y e^{-tX} - Y}{t} \tag{16}$$

It is easy to check that this is in fact antisymmetric and gives rise to a bilinear form on $\mathrm{Vect}(M)$

$$[X, Y] := L_X Y \tag{17}$$

A vector space endowed with such a bilinear form and satisfying the Jacobi identity is a **Lie algebra**.

Most important is when M itself has group structure, so is a **Lie group**, which we will denote by G. Then the vector fields on G of course also form a Lie algebra, just by virtue of the manifold structure of G.

We state the following proposition without proof

Proposition 16. *Let $\varphi : M \to N$ be a diffeomorphism of Lie groups. Then $\varphi_* : \mathrm{Vect}(M) \to \mathrm{Vect}(N)$ is a homomorphism of Lie algebras.*

For a Lie group, group elements induce automorphisms on the manifold by left multiplication, denoted L_g and by right multiplication R_g:

$$\begin{aligned} R_g &: G \to G,\ g : h \mapsto gh \\ L_g &: G \to G,\ g : h \mapsto hg \end{aligned} \tag{18}$$

We have that each group element induces (by pushforward) a map between tangent spaces

$$\begin{aligned} (L_g)_* &: T_h G \to T_{gh} G \\ (R_g)_* &: T_h G \to T_{hg} G \end{aligned} \tag{19}$$

A vector field X is left-invariant if $(L_g)_* X(h) = X(gh)$.

By the proposition, we get that $(L_g)_*[X, Y] = [(L_g)_* X, (L_g)_* Y]$ so these left-invariant vector fields in fact form a Lie algebra for the group. It exactly the vector fields representing the symmetries of G.

In local coordinates, the commutator can be written as:

$$\begin{aligned} X =& X^\mu \partial_\mu, \ Y = Y^\mu \partial_\mu \\ [X,Y] =& (X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu)\partial_\mu \end{aligned} \tag{20}$$

Left-invariant vectors flow consistent with the group action:

$$(L_g)_* X(e) = X(g) \tag{21}$$

The set of all left-invariant vector fields can be uniquely extracted from their value at the identity by this rule, and in fact for any vector $x \in T_e G$, there is a corresponding left-invariant vector field $X(g) = (L_g)_* x$. Therefore the tangent space to the identity gives rise to a Lie algebra which we will call the Lie algebra of G and denote by $\mathfrak{g}$. This Lie algebra (often referred to as *the* Lie algebra $\mathfrak{g}$ associated to the group G) is finite dimensional when G is.

Now because we define the Lie algebra as the "tangent space to the identity", it is worth asking "how does the Lie algebra appear at a generic point, g, on the group?". The idea is to bring that vector back to the identity using G and see what it looks like.

This is accomplished by using the **Maurer-Cartan form** Θ, which is a $\mathfrak{g}$-valed 1-form on G so that

$$\Theta(g) = (L_{g^{-1}})_* \tag{22}$$

Note that this maps from $\text{Vect}(G) \to \mathfrak{g}$. It takes a vector v at point g and traces it back to the natural vector at the identity that would have gotten pushed forward to v under g.

Proposition 17 (Properties of exp)**.** *For G a compact and connected Lie group, with Lie algebra $\mathfrak{g}$, we have a map* $\exp : \mathfrak{g} \to G$.

1. $[X,Y] = 0 \Leftrightarrow e^X e^Y = e^Y e^X$
2. *The map $t \to \exp(tX)$ is a homomorphism from $\mathbb{R}$ to G.*
3. *If G is connected then* exp *generates G as a group, meaning all elements can be written as some product $\exp(X_1)\ldots\exp(X_n)$ for $X_i \in \mathfrak{g}$*
4. *If G is connected and compact then* exp *is surjective. It is almost never injective.*

Example 18. The Lie algebra associated to the Lie group $\text{U}(n)$ of unitary matrices is $\mathfrak{u}(n)$ of antihermitian matrices. This is the same as the Lie algebra for the group $\text{SU}(n)$

Definition 19 (Adjoint Action on G)**.** For each g we can consider the homomorphism $\text{Ad}_g : h \mapsto ghg^{-1}$ or $\text{Ad}_g = L_g \circ R_{g^{-1}}$. This defines a representation

$$\text{Ad} : g \to \text{Diff}(G) \tag{23}$$

Definition 20 (Adjoint Representation of $\mathfrak{g}$)**.** The pushforward of this action gives rise to the **adjoint representation** of the Lie group $\mathfrak{g}$ by

$$(\mathrm{Ad}_g)_* = (L_g \circ R_{g^{-1}})_* \tag{24}$$

From the product rule, this acts as $[g, -]$ at the identity. We denote this as

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}\ \mathfrak{g} \tag{25}$$

The Jacobi identity ensures that ad is a homomorphism. If the center of G is zero then ad is faithful and we have an embedding into $\mathrm{GL}(n)$. This is nice because it also shows that modulo a central extension, every Lie algebra can be represented into $\mathrm{GL}(n)$, a weaker form of Ado's theorem.

Moreover the adjoint representation gives rise to a natural metric on G called the **Killing Form** given by

$$\kappa(X, Y) = \mathrm{Tr}(\mathrm{ad}(X)\mathrm{ad}(Y)) \tag{26}$$

3 Associated Bundles

Take a principal bundle $P(M, G)$ and let F be a space with associated automorphism $\mathrm{Aut}(F)$ so that $\rho : G \to \mathrm{Aut}(F)$ is a faithful representation. Then $g \cdot f$ is a well-defined notion, with free action, and we can consider the (right) action of G on $P(M, G) \times F$ by

$$g \cdot ([p, h], f) = ([p, hg], \rho(g)^{-1} f) \tag{27}$$

This is a free action as well. Then if G is compact (important) we have the orbit space

$$E_\rho = P(M, G) \times F/G \tag{28}$$

is a manifold

Theorem 21. *The space E_ρ can be made into a fiber bundle over M with fiber F called the **associated fiber bundle** of $P(M, G)$.*

Proof. We make $P \times F$ into a bundle by defining the projection

$$\pi_\rho([p, h], f) = p \tag{29}$$

and trivializations $\psi_\alpha : U_\alpha \times F \to \pi^{-1}(U)\alpha$ by

$$(\psi_\rho)_\alpha(p, f) = ([p, s_\alpha(p)], f) \tag{30}$$

and inverse

$$(\psi_\rho)_\alpha^{-1}([p, g], f) = [p, \rho(g) f] \tag{31}$$

□

From this, if F is a group then we can make $\pi_\rho^{-1}(p)$ into a group at each fiber in the obvious way, defining $[(p, v)][p, w] = [p, vw]$. And if F is a vector space then we can do the same construction to make each fiber have vector space structure.

The two associated bundles that we'll care about are $P(M, G) \times_{\mathrm{Ad}} G$ and $P(M, G) \times_{\mathrm{ad}} \mathfrak{g}$.