Factorization Structures

Goal: Understand $D(Bun_G)$

Recall $D(Bun_G) \hookrightarrow D(Bun_G^{B-\text{gen}})$

Fully faithful

One can show $D(Bun_G^{B-\text{gen}}) \hookrightarrow D(Bun_G^{U-\text{gen}})$

Question: Why is $D(Bun_G^{U-\text{gen}})$ easier given that $Bun_G^{U-\text{gen}}$ is NOT Artin stack in general?

$BG$ is Artin;

$G \times G \times G \xrightarrow{i} G \times G \xrightarrow{\pi} G \rightarrow 1$

"colimit of affine derived schemes w/ smooth morphisms"

Answer: $\exists$ Whitt (G) $\rightarrow D(Bun_G^{U-\text{gen}})$

whittaker structure

factorization

Whittaker coefficient
1) Factorization algebras

\[ \text{Gr}_\mathcal{G}^\mathrm{k} \]

affine Grassmannian

\[ x \in \mathcal{C} \]

In terms of functors of points

\[ x : \mathcal{C} \to \mathcal{C} \]

\[ \text{Gr}_\mathcal{G}^\mathrm{k}(s) \]

\[ = \left\{ \text{\begin{align*}
\text{\ G-bundles on} & \ G_s = \mathcal{C} \times S \\
\text{w/ trivialization on} & \ (G_s \times _\mathcal{C} \mathcal{S}) \times S \\
\text{on} & \ \mathcal{C} \times S \end{align*}} \right\} \]

Recall previously \( \text{Gr}_\mathcal{G}(\mathcal{R}) = \mathcal{G}(k((t))) / \mathcal{G}(k[[D^\pm]]) \)

Define \( \text{Gr}'_{\mathcal{G},x} \) by:

\[ \text{Gr}'_{\mathcal{G},x} := \left\{ \text{\begin{align*}
\text{\ G'-bundles on} & \ ID_s \\
\text{w/ trivialization on} & \ ID_s^x \\
\text{on} & \ \mathcal{S} = \text{Spec} \ k \\
ID = \text{Spec} \ k[[t]] & \ ID_s := \text{Spec} \ A[[t]] \\
ID^x = \text{Spec} \ k((t)) & \ ID_s^x = \text{Spec} \ A((t))
\end{align*}} \right\} \]

Then (Beauville - Laszlo)

\[ \text{Gr}_\mathcal{G}^\mathrm{k} \to \text{Gr}'_{\mathcal{G},x} \]

is an isomorphism
\[ \Rightarrow \text{Gr}_{G,x}(k) = \text{Gr}_{G,x}(\text{Spec } k) \]
\[ = \mathbb{G}(k((t))) \backslash \mathbb{G}(k[[t]]) \]

**Idea of Beilinson - Drinfeld:**

- can think about
- \( n \times n \) matrix
- \( G_{\text{affine}} = \left\{ \begin{pmatrix} x_1 & \cdots & x_n \\ \cdots & \cdots & \cdots \\ x_n & \cdots & x_1 \end{pmatrix} : s \to C \right\} 
- \( G \)-bundle on \( C \)
- \( L \)-bundle on \( C \)

**Look at** \( n=2 \)

- \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \text{Gr}_{C,2} \)
- \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \mathbb{C}^2 \)

1. if \( x = y \), it is \( \text{Gr}_{G,x} \)

2. if \( x \neq y \), it is \( \text{Gr}_{G,x} \times \text{Gr}_{G,y} \)

**Heuristic:** \( PG \) \( G \)-bundle on \( C \) w/ trivialization on \( C(\mathbb{K}) \)

- \( P \) on \( C(\mathbb{K}) \)
- \( P' \) on \( C(\mathbb{K}) \)
- \( P \) on \( C(\mathbb{K}) \)

**For** \( I \to J \) surjective map of finite sets

\[ A : C^J \to C^I \]

\[ (C_j)_{j \in J} \to (C_i)_{i \in I} \]

\[ C_j \to C_i \]
(2) \[ \text{Factorization} \]
\[
\begin{array}{c}
\text{Gr}_{E, \mathcal{C}^I} \\
\times (\mathcal{C}^I \times \mathcal{C}^I)_{\text{dij}} \\
\downarrow \\
\text{Gr}_{E, \mathcal{C}^I} \\
\end{array}
\]
\[ I = I_1 \sqcup I_2 \]

where \( \mathcal{C}^I \times \mathcal{C}^I_{\text{dij}} = \delta_{ij} \mathcal{C}_i \times \mathcal{C}_i \)

\[ \forall i \in I_1, \forall j \in I_2 \]

**Remark:** \( \text{Gr}_{E, \mathcal{C}^I} \xrightarrow{\text{formally smooth}} \mathcal{C}^I \)

ind-scheme of
ind-finite type
ind-proper for \( E \) reductive

**Definition:** D-space over \( X \) (CDx-space)
is an object of \( \text{PreStk}/X_{\text{dR}} \)

\[
\text{Gr}_{E, \mathcal{C}^I} \rightarrow \text{Gr}_{E, \mathcal{C}^I}_{\text{dR}} \leftarrow \text{D-space over } \mathcal{C}^I
\]

because

\[ X_i : S \rightarrow C \]

\[ \otimes X_{i \text{red}} : S_{\text{red}} \rightarrow C \]
**Defn 1** A **factorization space** over $C$ is an assignment

\[ I \rightarrow y_i \in \operatorname{Pre Stk}_{/C_{dR}}^I \]

satisfying the Ran axiom and the factorization axiom

**Ex 1** $C_{dR}$ is a fact. space

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**Factorization algebra**

~linearization of factorization space

**Defn 1** A **factorization algebra** is an assignment

\[ I \rightarrow A_{c I} \in D(C^I) = QC(C_{dR}^I) \]

s.t.

1. \( \forall I \rightarrow J \ \Delta f: C^I \rightarrow C^J \)

\[ \Delta^I f \ A_{c I} \sim A_{c J} \]

2. (Factorization)

\[ A_{c I} \left( (c^I \times c^I)_{\text{disj}} \right) = \left( A_{c I} \otimes A_{c I_2} \right) \]  

\[ \text{for } I = I_1 \cup I_2 \]

For \( \mathcal{F} \in D(CX) \), \( \mathcal{G} \in D(CY) \)

\[ \mathcal{F} \otimes \mathcal{G} = \pi_1^* \mathcal{F} \otimes \pi_2^* \mathcal{G} \]

where \( \pi_1: X \times Y \rightarrow X \)

\( \pi_2: X \times Y \rightarrow Y \)
Example

$I \mapsto w_c$ is a factorization algebra

[Recall $X \xrightarrow{\rho^x} pt$, $w_x := \rho^x_k$]

1. $C^j \xrightarrow{\Delta} C^1$

2. $w_{c_1} \otimes w_{c_2} = w_{c_1}$

More generally, given a factorization space $\Sigma y_i$ over $C$, one can construct a factor algebra $A_{\Sigma y_i} = \prod_i \mathcal{D}(\mathbf{x}_i \otimes y_i)$ where $\pi_i : y_i \to C^1$ is nice enough provided $y_i$ is nice enough, e.g., $y_i$ is ind-scheme of ind-finite type.

Summary

Fact:

$A(x_1 \cdots x_n) \simeq A_{x_1} \otimes \cdots \otimes A_{x_n}$

if $x_i \neq x_j$ then $u_{ij}$

$\text{Ran} \Rightarrow A$ depends only on $x_1, \ldots, x_n \subset C$

subset
all information is in \( x \leftarrow I = \text{Sp}^3 \)
together with collision data

Big picture (Interlude)

\[
\begin{align*}
1) & \quad \text{Bun}_G^{\text{N-gen}} \\
\downarrow & \\
\text{Fun} (\mathbf{M}(k) \setminus G(\mathbf{A})/G(0)) \\
\downarrow & \\
\text{Field} & \\
\Downarrow & \\
\text{Gr}_c^{2} & \leftarrow G(\mathbf{A})/G(0)
\end{align*}
\]

\( k = k(\mathcal{L}) \)

2) Group actions on Categories

i) Sheaves of Categories (??)

\( \text{shvCat/y} \) for a prestack \( y \)

\( \text{affine derived scheme} \)

\( \text{shvCat/s} = \text{QC}(s) - \text{mod}(\text{DGCat}) \)
$QC(s) = (A \text{-mod, } \otimes)$

is a comm alg obj in $DG \text{Cat}$

classified: $A \in Alg = Alg(\text{Vect}) \triangleleft \exists m : A \otimes A \to A$

$M \in A \text{-mod} = A \text{-mod}(\text{Vect}) \triangleleft A \otimes M \to M$

now: $A \text{-mod} \in Alg(DG \text{Cat}) \leftarrow \mathcal{F} \in (A \text{-mod})^{-\text{mod}}(DG \text{Cat})$

$A \text{-mod} \otimes \mathcal{F} \to \mathcal{F}$

$\mathcal{F} \in (A \text{-mod})^{-\text{mod}}(DG \text{Cat})$

$\mathcal{F} \in \mathcal{H}C(\mathcal{F})$

End (id: $\mathcal{F} \to \mathcal{F}$)

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**Def (ShvCat)$/y = \lim_{S \to y} \text{ShvCat} / S$$

$\text{ShvCat} / S \to \mathcal{F}^* \text{ShvCat} / S$

$\to QC(y) \text{-mod } \triangleleft \mathcal{C} \to QC(S) \otimes QC(\mathcal{C}) \triangleleft$

$\Gamma : \text{ShvCat} / y \to DG \text{Cat}$

$\Gamma : \text{QC}(y) \to \text{Vect}$

$\Gamma (y, F) = \lim_{S \to y} \Gamma (S, \mathcal{F}^* F)$

$\mathcal{F} \to \mathcal{F}(y) \text{-mod } \lim_{S \to y} (S, \mathcal{F}^* F)$
\[
\text{Shv/Cat}/_{/Y} \cong \text{DG Cat}
\]
\[
\nu_{/Y} \iff \text{QC}(S)
\]
\[
\in \text{QC}(S) - \text{mod}(\text{DG Cat})
\]
\[
\Rightarrow \text{QC}(S)^{\times 2} \otimes \text{QC}(Y)
\]
\[
\nu_Y \iff \sum_{s \in S} \frac{1}{s+y}
\]
\[
\Rightarrow \nu_Y
\]

Recall \( Y \) is called \( t \)-affine if \( \Gamma : \text{QC}(Y) \rightarrow \text{QC}(Y)^{\times 2} \) is an equivalence (Gaitsgory).

**Def:** \( Y \) is called \( t \)-affine if \( \Gamma : \text{QC}(Y) \rightarrow \text{QC}(Y)^{\times 2} \) is an equivalence.

**Ex:** quasi-separated quasi-compact schemes
- Artin stacks of almost finite type
- For \( S \) of finite type, \( S_{/Y} \)

**Non-example:**
\[
\mathcal{A}^\infty = \lim_{n} \mathcal{A}^n
\]
Definition: $G\text{-cat} := \text{shvCat}_{/BG_{dR}}$

why? $G\text{-rep} = \text{Rep}_G = QC(BG)$

$(P, V) \mapsto V$ underlying vector space

$V^G$ invariants

$(BG \to \text{pt})$

$\Gamma: QC(BG) \to \text{Vect}$

$i: (P, V) \mapsto \Gamma(BG, V) = V^G$

$\pi: \text{pt} \to \text{pt}/G$

$QC(\text{IBG}) \xrightarrow{\pi^*} \text{Vect}$

$i: (P, V) \mapsto V$

$\text{shvCat}_{/BG_{dR}} \to \text{DGCat}$

$\xi \mapsto \Gamma(\text{pt}, \pi^* \xi)$

$G$

$\xi$

$\Gamma$ $\text{shvCat}_{/BG_{dR}} \to \text{DGCat}$

$\xi \mapsto \Gamma(BG_{dR}, \xi)$

$BG_{dR}$ is not 1-affine