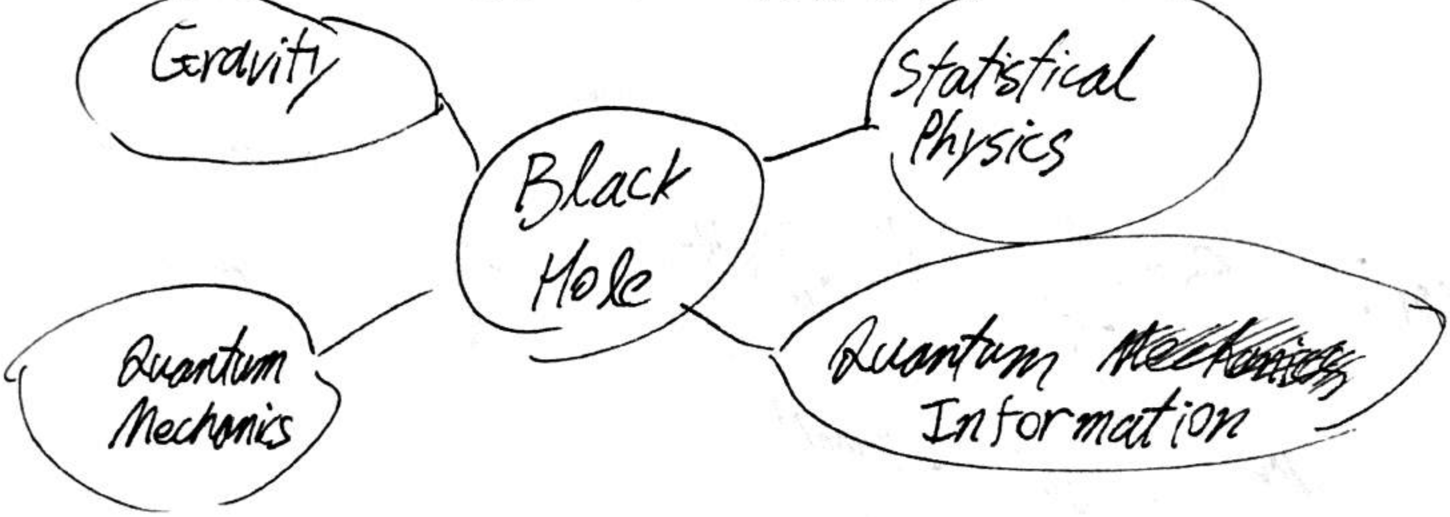
Chapter 1:
Black Holes & the Holographic Principle

BHs: 1) Key object – astrophysically ubiquitous

2) Quantum Matter around BH
→ Hawking's 1975 paper
↝ Black Hole radiation



↝ Black holes bring Quantum Gravity to a Macroscopic level.

1.1 General Remarks on Gravity

all other interactions: probed to 10^{-33} cm (Large Hadron Collider)
gravity: only 10^{-2} cm

General Relativity: "gravity = spacetime"

Quantum Gravity: " = quantum spacetime"

Question: What is ~~an~~ the relationship between quantum gravitational effects and the nature of spacetime?

Answer: A) Einstein Gravity & Gravitons

line element: $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$

Einstein's Equations: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R - 2\Lambda) = 8\pi G_N T_{\mu\nu}$ (*)

Λ ↑ cosmological const.

$T_{\mu\nu}$ ↑ matter (i.e. stress-energy tensor)

Action: $S = \frac{1}{16\pi G_N} \int d^d x \sqrt{-g}\,(R - 2\Lambda) + S_{matter}$ (**)

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}}$$

take $\underline{\Lambda = 0,\ T_{\mu\nu} = 0}$

simplest solution to (*) → $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$

"Weak gravity" $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ (1) with $\kappa^2 = 8\pi G_N$

putting (1) into (**), we get

$$S = \int d^4x \left[\mathcal{L}_2 + \kappa \mathcal{L}_3 + \kappa^2 \mathcal{L}_4 + \cdots \right] + \frac{\kappa}{2} h_{\mu\nu} T^{\mu\nu} + O(\kappa^2)$$

canonically normalized

starts at $O(\kappa^2) \simeq O(h^2)$ since $\eta_{\mu\nu}$ solves EOM

cancels $16\pi G_N$

EOM from $\mathcal{L}_2$ give plane wave solutions
These are what we call "gravitational waves"

$\mathcal{L}_2$ is quadratic in $h \Rightarrow$ free field theory for h
$\hookrightarrow$ quantize $\mathcal{L}_2 \Rightarrow$ spin-2 massless particle "graviton"

$\rightsquigarrow \mathcal{L}_3, \mathcal{L}_4, \ldots \rightarrow$ self-interaction of the graviton

gravitational interaction of matter is by exchange of an $h_{\mu\nu}$
e.g. electron as matter has $T_{\mu\nu} \sim \bar{\psi}\psi$

$\Rightarrow$ 

Treated as QFT, S is non-renormalizable.
$\rightsquigarrow$ So we can treat $S[h]$ as an effective field theory not fundamental.

B) Important Scales of Gravity

Planck scales: $\hbar, G_N, c=1 \Rightarrow$
$M_p = \sqrt{\frac{\hbar}{G_N}} \simeq 1.2 \cdot 10^{19}\,\text{GeV}$
$l_p = \frac{\hbar}{M_p} = \sqrt{\hbar G_N} \simeq 1.6 \cdot 10^{-33}\,\text{cm}$
$t_p = l_p \simeq 5.4 \cdot 10^{-44}\,\text{sec}$

For reference: top quark mass: $\approx 10^{-17} M_p$
electron mass: $\approx 10^{-23} M_p$

Largeness of $M_p \leftrightarrow$ weakness of gravity at microscopic scales
$\Rightarrow$ Consider two particles of mass m brought to nearest possible distance $r_c = \hbar/m$ "Compton wavelength"

$$\rightsquigarrow \lambda_G := \frac{V_g(r_c)}{m} = \frac{1}{m}\frac{G_N m^2}{r_c} = \frac{G_N m^2}{\hbar} = \frac{m^2}{M_p^2}$$

We see $\lambda_G = \frac{m^2}{M_p^2} = \frac{l_p^2}{r_c^2}$. For $m \ll M_p$, $\lambda_G \ll 1$ e.g. $e^- \to \lambda_G \sim 10$

For $m \sim M_p$, $\lambda_G \sim O(1) \Rightarrow$ Quantum Gravity

Relativistic calculation: $\lambda_G \sim E^2/M_p^2$, E c.o.m. energy

If we take $m \gg M_p$, does $\lambda_G \gg 1$? No!

Different question: take point particle of mass m. At what distance r_s from it does classical gravity become strong?

probe $m' \to \frac{\frac{G m m'}{r_s}}{m'} \sim 1 \Rightarrow r_s = G_N m$

Remarks: 1) In Newtonian gravity, at r_s escape velocity $\sim c$

2) In GR, $r_s \sim$ Schwarzschild radius of Black Hole

$\Rightarrow r_s \sim$ minimal distance we can probe an obj. in classical gravity.

Two important scales: $r_c \sim \hbar/m$, $r_s \sim G_N m$ $\Rightarrow \frac{r_s}{r_c} = \frac{G_N m^2}{\hbar} = \left(\frac{m}{M_p}\right)^2$

1) $m \ll M_p \Rightarrow r_s \ll r_c$ so Compton wavelength outside r_s

$\Rightarrow$ gravitation is weak, negligible [r_s not important quantum effects dominate

2) $m \sim M_p$, $r_s \sim r_c$, $\lambda_G \sim 1$ Quantum Gravity becomes important

3) $m \gg M_p$, $r_s \gg r_c \Rightarrow r_c$ not relevant $\Rightarrow$ Classical gravity dominates

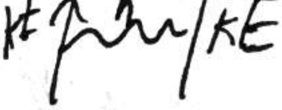
The relationship between Black Holes and quantum gravity, however, affects much more than Planck scale physics.

4]

Last time.

$$\lambda_G \sim \frac{G_N E^2}{\hbar} \sim \frac{E^2}{M_p^2}$$

$$\kappa h_{\mu\nu} T^{\mu\nu}, \quad \kappa^2 = 8\pi G_N$$



$$\frac{r_s}{r_c} \sim \frac{m^2}{M_p^2}$$

corollary: l_p is minimal localization length

non-grav: $\delta l \sim \hbar / E$

with gravity: $E \sim M_p$

$$r_s \sim G_N E \sim r_c \sim l_p$$

$$E \gg M_p$$
$$r_s \gg r_c$$

$$\delta p \sim \frac{\hbar}{\delta \lambda} \Rightarrow \delta \lambda > G\,\delta p \sim \frac{G\hbar}{\delta x}$$
$$\rightarrow \delta x^2 > \sqrt{G\hbar} \sim l_p$$

$E \ll M_p$, ignore: (1) grav. interaction
(2) fluctuations of $g_{\mu\nu}$ $\Rightarrow$ QFT in rigid spacetime (can be curved) i.e. on earth

BH

~~Math~~ Mathematical Treatment:

E fixed, $\hbar$ fixed, $G_N \to 0$

$(l_p \to 0,\ M_p \to \infty)$

C low energy expansion in G_N

$$Z = \int Dg\, D\Psi\, e^{iS[g,\Psi]}$$

$$S = \frac{1}{16\pi G_N} S_{grav}[g] + \frac{1}{\lambda} S_m[g,\Psi]$$

λ: matter coupling

$\lambda \gg G_N$

$G_N \to 0 \Rightarrow$ saddle point: $\delta S_{grav}[g] = 0$

$\Rightarrow g_{classical}$

Expand $g = g_{classical} + \kappa h$

$$\Rightarrow S = \frac{1}{16\pi G_N} S_{grav}[g_c] + \underbrace{\frac{1}{\lambda} S_m[\Psi, g_c]}_{\text{QFT in curved spacetime}} + S_2[h] + \cdots + \kappa h_{\mu\nu} T^{\mu\nu}$$

small G_N expansion breaks down at $\frac{E^2}{M_p^2} \sim O(1)$.

For a sphere of radius L, p is quantized as $\frac{1}{L}$

$$\Rightarrow E^2 \sim p^2 \sim \frac{1}{L^2} \sim R$$

$\frac{G_N E^2}{\hbar}$

D. gravity in general dimensions

$$S_{grav} = \frac{1}{16\pi G_d} \int d^d x \sqrt{-g}\,(R - 2\Lambda)$$

$$[G_d] = \frac{L^{d-1}}{MT^2} \Rightarrow M_{pd}^{d-2} = \frac{\hbar^{d-3}}{G_d}, \quad l_{pd}^{d-2} = \hbar G_d$$

$$\lambda_G \sim \frac{G_d E^{d-2}}{\hbar^{d-3}} \sim \frac{E^{d-2}}{M_{pd}^{d-2}}$$

$$r_s \sim (G_N m)^{\frac{1}{d-3}}$$

consider $M_d = M_D \times Y$

M_D non-compact, D-dimensional

Y compact, $d-D$

suppose Y is too small to be detected.

$R \times S^1$

The effective Newton constant G_D for an observer is not the same as the fundamental

$$\frac{1}{G_D} = \frac{1}{G_d} V_Y \leftarrow \text{volume of } Y$$

$$l_{pD}^{D-2} = \frac{l_{pd}^{d-2}}{L^{d-D}}, \quad \text{expect } L > l_{pd}$$

$$\Rightarrow l_{pD} < l_{pd}$$

E. Einstein gravity as E.F.T.

gravity tested to 10^{-2} cm

we're going down to 10^{-33} cm

1) Extra dimensions

will see d-dim gravity before reaching l_p

2) string theory

l_s string length

3) Suppose new physics appears at some scale $l \sim \frac{1}{M}$

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \left[R - 2\Lambda + \frac{a_1}{M^2} R^2 + \frac{a_2}{M^2} R_{\mu\nu} R^{\mu\nu} + \dots \right]$$

1.2 | Classical BH Geometry I

consider a spherically symmetric, electrically neutral object of mass M

The Schwarzschild solution (4D) can be analytically found to be:

$$ds^2 = -f\,dt^2 + \frac{1}{f}\,dr^2 + r^2\,d\Omega^2$$
$$= g_{\mu\nu}\,dx^\mu dx^\nu$$

with $f = 1 - \frac{2G_N M}{r} = 1 - \frac{r_s}{r}$

$r_s := 2G_N M$

Most important features:

1) $r \to \infty$, $f \to 1$, $g_{\mu\nu} \to \eta_{\mu\nu}$

2) $r = r_s$, $f \to 0$, $g_{tt} = 0$
$g_{rr} = \infty$

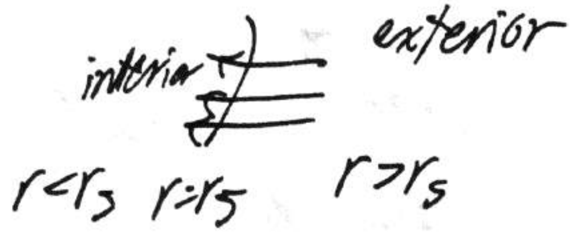
will see t (Schwarzschild time) becomes singular at $r = r_s$

$r = \text{const} > r_s \Rightarrow$ time-like hypersurface

$r = \text{const} < r_s \Rightarrow$ space-like hypersurface

$r = r_s \Rightarrow$ null hypersurface

3) $r = r_s$: event horizon



4) $r = r_s$: hypersurface of infinite redshift

consider an observer O_h at $r = r_h \gtrsim r_s$

— — — — O_∞ at $r = \infty$

proper time for O_∞: t

proper time for O_h

$$d\tau_h = f^{1/2}\, dt = \left(1 - \frac{r_s}{r_h}\right)^{1/2} dt$$

consider a physical process at $r = r_h$
with local proper energy ε

O_∞ sees energy $E_\infty = \varepsilon \left(1 - \frac{r_s}{r_h}\right)^{1/2}$

as $r_h \to r_s$, $E_\infty \to 0$ "infinite redshift"

$$\boxed{ds^2 = -f\, dt^2 + \frac{1}{f} dr^2 + r^2 d\Omega_2^2}$$

$f = 1 - \frac{r_s}{r}$, $r_s = 2GM$

causal structure & Rindler spacetime

consider $r \gtrsim r_s$ $\frac{r - r_s}{r_s} \ll 1$

proper distance ρ from $r = r_s$

s.t. $d\rho^2 = \frac{dr^2}{f} \Rightarrow d\rho = \frac{dr}{\sqrt{f}}$

$$f(r) = \cancel{f(r_s)} + f'(r_s)\underbrace{(r - r_s)}_{\frac{dr}{f}} + \cdots$$

$$\Rightarrow \rho = \frac{2}{\sqrt{f'(r_s)}} \sqrt{r - r_s}$$

$$\Rightarrow f(r) = \left[\frac{1}{2} f'(r_s)\right]^2 \rho^2 = \kappa^2 \rho^2$$

$$\Rightarrow ds^2 = -\kappa^2 \rho^2 \underline{dt}^2 + d\rho^2 + r_s^2 d\Omega^2$$

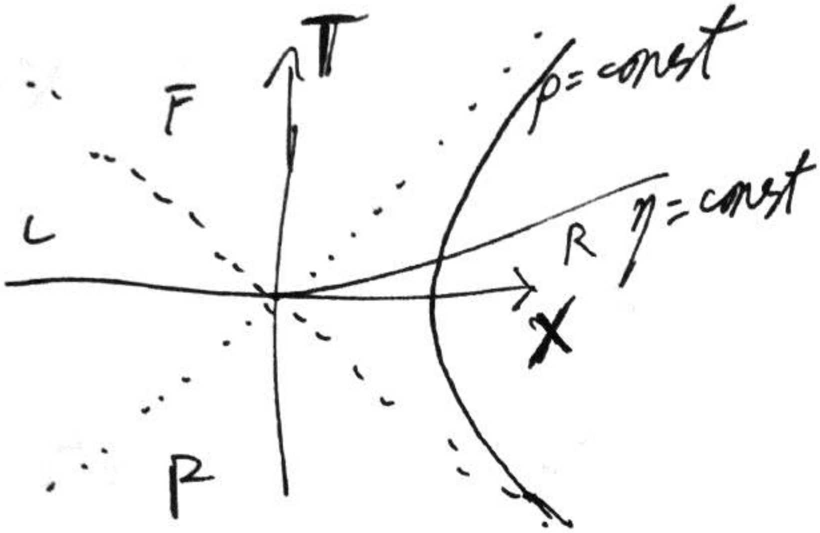
$$= \underbrace{-\rho^2 d\eta^2 + d\rho^2}_{\text{Mink}_2 \text{ (Rindler spacetime)}} + r_s^2 d\Omega^2$$

$\eta = \kappa t$

$$ds^2 = -dT^2 + dX^2 = -\rho^2 d\eta^2 + d\rho^2$$

$\Downarrow$

$X = \rho \cosh\eta$ $T = \rho \sinh\eta$



$\rho^2 = X^2 - T^2$

$\tanh\eta = T/X$

$$X = T = 0 \Rightarrow \rho \to 0,\ \eta \text{ finite}$$

~~At~~

$$X = T \Rightarrow \rho \to 0,\ \eta \to +\infty \text{ s.t. } \rho e^{\eta} \text{ finite}$$

$$X = -T \Rightarrow \rho \to 0,\ \eta \to -\infty \text{ s.t. } \rho e^{-\eta} \text{ finite}$$

Rindler observers: $\rho = \text{const}$ ($\Rightarrow r = \text{const}$)

$$\Rightarrow a_{prop} = 1/\rho$$

Note: No signal can propogate from F to R

$X = T$: future horizon (can only go in)

$X = -T$: past horizon (can only come out)

1) $r = r_s \leftrightarrow \rho = 0 \leftrightarrow X = \pm T$

null hypersurface

(ρ, η) singular at $\rho = 0 \Leftarrow (t, r)$ singular at $r = r_s$

2) $r=$ const observer

$\Leftrightarrow \rho =$ const rindler observer

their accelerations agree

3) free-fall observer in BH $\Leftrightarrow$ inertial observer in Rindler
Minkowski

4) Using (T, X), we can extend the black hole geometry from $r \gg r_s$ to <u>four regions</u> with the near-horizon metric

$$ds^2 = -dT^2 + dX^2 + r_s^2\, d\Omega^2$$

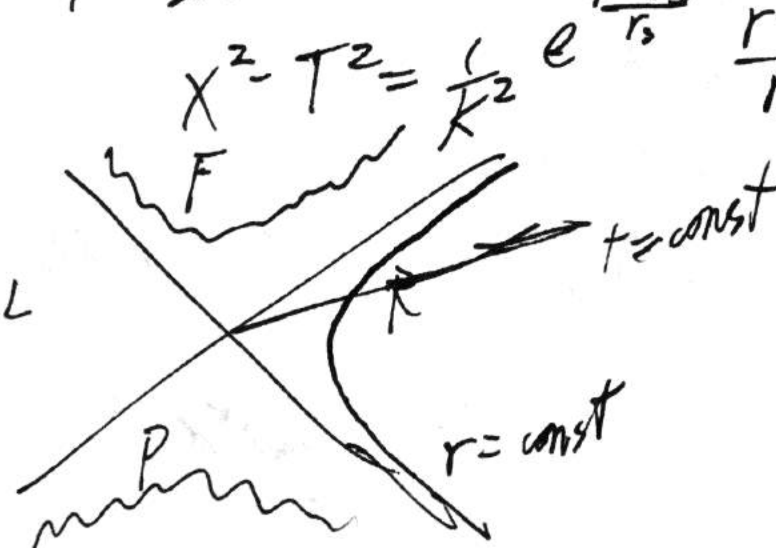
T, X as coordinate transform of (r,t) and then extend ~~a~~ them to <u>full spacetime</u>

(derive) $$ds^2 = g(r)(-dT^2 + dX^2) + r_s^2\, d\Omega_2^2$$

$$g(r) = \frac{r_s}{r} e^{-\frac{r-r_s}{r_s}}$$

r should be considered as a function of (X, T)

$$X^2 - T^2 = \frac{1}{k^2} e^{\frac{r-r_s}{r_s}} \frac{r-r_s}{r_s}$$



a) $g(r_s) = 1$

$X^2 - T^2 = 0$
$(r=r_s)$

b) singularity at $r=0$

$\Leftrightarrow T^2 - X^2 = \frac{1}{k^2} > 0$

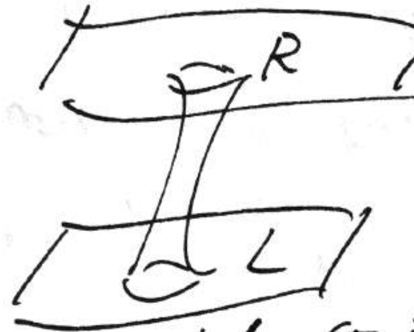
c) symmetries

(i) $T \leftrightarrow -T \qquad X \leftrightarrow -X$

(ii) boost in $(T, X) \Leftrightarrow t \to t + \text{const}$

d) L is a mirror of R w/ another asymptotic flat region

e) $T=0$ slice



wormhole (E-R)
non-traversible

f) F: interior of BH (Future horizon)

g) P: white hole (past horizon)

h) L, P not present in collapse of a star

A digression: Penrose diagrams

Procedure. 1. choose a metric
$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$
x^μ covers full spacetime

2. Find $x^\mu = x^\mu(y^\alpha)$ s.t. y^α has finite range
$$ds^2 = g'_{\alpha\beta}\, dy^\alpha dy^\beta$$

3. construct a new metric
$$\tilde{ds}^2 = \Omega^2(y)\, ds^2 = \tilde{g}_{\alpha\beta}\, dy^\alpha dy^\beta$$

so that the causal structure of $\tilde{g}$ is known

Mink$_2$: $ds^2 = -dT^2 + dX^2 \qquad U = T - X$

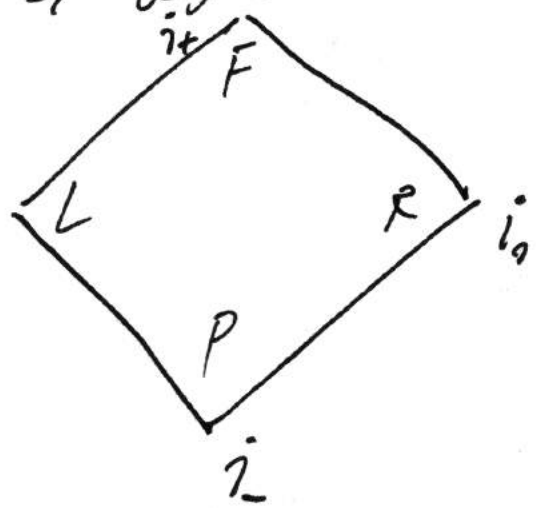
$= -dU\,dV \qquad V = T + X$

$U = \tan u \Rightarrow u, v \in [-\pi/2, \pi/2]$

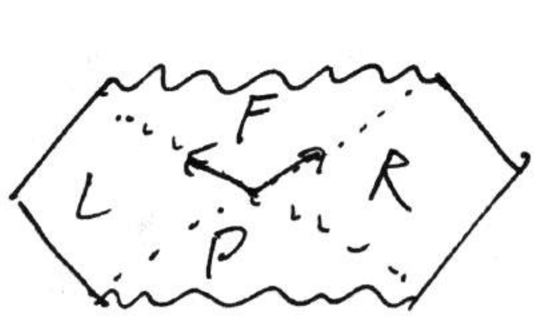
$V = \tan v \qquad ds^2 = -\dfrac{1}{\cos^2 u \cos^2 v}\,du\,dv$

$\Rightarrow \widetilde{ds}^2 = -du\,dv$

$\Rightarrow$

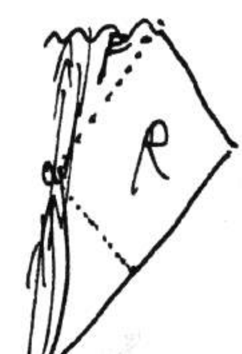


Black Hole:



Kruskal coordinates (X, T)

steller collapse:



Formulas we will use going forward

$$ds^2 = -f\,dt^2 + \frac{1}{f}dr^2 + r^2 d\Omega^2$$

$$= -\kappa^2\rho^2 dt^2 + d\rho^2 + r_s^2\, d\Omega^2 + \dots \leftarrow \text{near horizon}$$

$$\kappa = \frac{1}{2} f'(r_s) = \frac{1}{2r_s} = \frac{1}{4G_N M}$$

1.3 Black Hole temperature

1975: Hawking
1976: Unruh
Bisognano – Wichmann

Both phenomena at level of leading order in low-energy approx

QFT in a rigid curved spacetime

This effect is universal insofar as it would apply to any QFT regardless of interactions of matter content.

e.g. (*) $S = -\int d^4x \sqrt{g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + V(\varphi) \right]$

1.3.1 Hawking & Unruh temperatures from Euclidean analytic continuation

$$\rho_\beta = \frac{1}{Z_\beta} e^{-\beta H}$$

with $Z_\beta = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\left(e^{-iHt/\hbar} \right)$

$t = -i\beta\hbar$

$ds^2 = -dt^2 + d\vec{x}^2$

$t \to -i\tau \qquad \tau \sim \tau + \beta\hbar \qquad ds_E^2 = d\tau^2 + d\vec{x}^2 \quad (1)$

Thermal equilibrium at $T = 1/\beta$ described by path integrals in (I) with periodicity $\beta\hbar$

(A) Hawking temp: take BH metric

$$t \to -i\tau$$

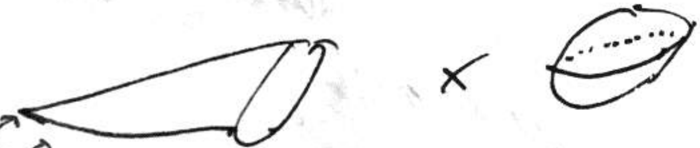
$$\begin{aligned} ds_E^2 &= f\,d\tau^2 + \frac{1}{f}dr^2 + r^2 d\Omega^2 \\ &= \kappa^2\rho^2 d\tau^2 + d\rho^2 + r_s^2 d\Omega^2 \\ &= \underbrace{\rho^2 d\theta^2 + d\rho^2}_{\text{locally } \mathbb{R}^2} + r_s^2 d\Omega^2 \end{aligned}$$

$\theta = \kappa\tau$ is like an angular variable

Global structure: depends on periodicity of θ

$\theta \sim \theta + 2\pi \Rightarrow$ globally $\mathbb{R}^2$

other periodicity:



$\rho = 0$

$\rightsquigarrow$ ALE? gravitational instantons

$\rho = 0 \Rightarrow$ conical singularity

smoothness of Euclidean geometry

$\Rightarrow \tau \to \tau + \frac{2\pi}{\kappa}$ (uniquely determined)

Different from on Mink.

$$R \times \mathbb{R}^3 \to S^1 \times \mathbb{R}^3$$

any period allowed

$\Rightarrow$ in a black hole geometry, quantum matter can be in equilibrium only at a single temperature $T_H = \dfrac{1}{\beta_H}$

$$\hbar\beta_H = \frac{2\pi}{\kappa} \Rightarrow T_H = \frac{\hbar\kappa}{2\pi} = \frac{\hbar}{8\pi G M}$$

Remarks:

1) T_H should be considered as temperature measured in ~~proper~~ units of proper time at $r = \infty$.

2) $\Rightarrow$ BH must have temperature T_H

3) Field theory on a cone
$\Rightarrow$ observables can be singular at the singularity.

4) suppose $\tau \sim \tau + \beta\hbar$ $\quad \beta \neq \beta_H$

must be singular to screen difference

$T \neq T_H$

$\rho = 0 \Leftrightarrow$ horizon

"force" the equilibrium

5) You can put any matter at any T outside the black hole (including nothing, $T=0$)

$\rightsquigarrow$ non-eq. state

but euclidean A.C. you can only desc. the equilibrium state.

6) $$ds^2 = g(r)\left(-dT^2 + dX^2\right) + r^2 d\Omega_2^2$$

$r = r(T^2 - X^2)$ $\quad\rightsquigarrow\quad T_E^2 + X^2 =$

$T \to -iT_E$

7) For a stationary observer at r

$$t_{loc} = \sqrt{f(r)}\, dt$$

$$\Rightarrow T_{loc} = \frac{\hbar k}{2\pi} f^{-1/2}(r) \quad (2)$$

$r \to r_s \qquad T_{loc} \to \infty$

B. Unruh temp

$$ds^2 = -dT^2 + dX^2$$

$\rightsquigarrow$ Rindler space $\quad ds^2 = -\rho^2 d\eta^2 + d\rho^2$

$\eta \to -i\theta$

$$ds_E^2 = \rho^2 d\theta^2 + d\rho^2$$

Smoothness of Euclidean space

$\Rightarrow \theta \sim \theta + 2\pi$

local time: $dt_{loc} = \rho\, d\eta$

$d\tau_{loc} = \rho\, d\theta$

$\tau_{loc} \sim \tau_{loc} + 2\pi\rho \; = \hbar\beta_{loc}$

$\Rightarrow T_u(\rho) = \dfrac{\hbar}{2\pi\rho} = \dfrac{\hbar a}{2\pi}$ (3), $a = 1/\rho$

$\Rightarrow$ a uniformly accelerated obs. in Mink can be in thermal equilibrium only at $T_u(\rho)$, otherwise one finds singular behavior at $T = \pm X$ $(\rho = 0)$

Remarks:

1) (2) and (3) agree when $r \simeq r_s$, as expected

2) BH: $r \to \infty$ $\quad T \to T_H$ $(a_{prop} \to 0)$

Rindler: $\rho \to 0$ $\quad T \to 0$ $(a_{prop} \to 0)$

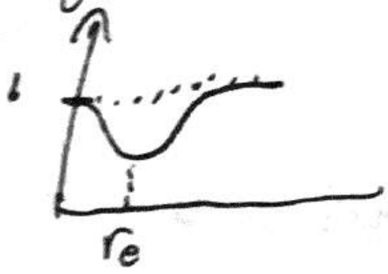
2) Does this happen to all accelerated observers?

$$ds^2 = -g(r)\,dt^2 + \frac{1}{g(r)}\,dr^2 + r^2 d\Omega_2^2$$

$$g(r) = 1 - \frac{2G_N m(r)}{r}$$

$$m(r) = \begin{cases} M_{earth} & r > r_e \\ \propto r^3 & r < r_e \end{cases}$$

$\Rightarrow$ $g(r)$



$t \to -i\tau$ τ can have any periodicity

Hawking, Unruh require $g_H \to 0$

3) Rindler $T \neq T_U$

singular behavior at $T = \pm X$

1.3.2 Unruh temp. from entanglement

1) Clarifies physical origin of temp.

2) Gives deeper understanding of the quantum state of matter

Ai digression — an alternative (Lorentzian) way to describe thermal states

$$\mathcal{H},\ H,\ E_n,\ |n\rangle,\ \rho = \frac{1}{Z_\beta} e^{-\beta H}$$

$\Rightarrow$ double the system

$$\mathcal{H}_{tot} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

$$\mathcal{H}_1 \simeq \mathcal{H}_2 \simeq \mathcal{H}$$

typical state:

$$\underbrace{\sum_{m,n} a_{mn}\, |m\rangle_1 \otimes |n\rangle_2}_{\text{non-factorizable}}$$

$$\neq \Psi_1 \otimes \Psi_2$$

$\Rightarrow$ entangled

normalized

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_n e^{-\frac{1}{2}\beta E_n} |n\rangle_1 \otimes |\tilde{n}\rangle_2 \qquad \langle \Psi_\beta | \Psi_\beta \rangle = 1$$

$|\tilde{n}\rangle$ is T-reversal of $|n\rangle$

$Z_\beta = \mathrm{Tr}(e^{-\beta H})$ ← either system

Consider X_1 which acts only on $\mathcal{H}_1$

$$\Rightarrow \langle \psi_\beta | X_1 | \psi_\beta \rangle = \frac{1}{Z_\beta} \sum_n e^{-\beta E_n} \langle n | X | n \rangle$$

$$= \mathrm{Tr}(\rho_B X)$$

$$\mathrm{Tr}_2 \left(|\psi_\beta\rangle \langle \psi_\beta| \right) = \rho_B$$

$|\psi_\beta\rangle$: thermal field double

Umezawa (1968?)

Remarks: Finite-T effects come from:

1) Special entangled structure of ψ_β
2) Ignorance of the other system
3) Purification of ρ_β
4) $(H_1 - H_2)|\psi_\beta\rangle = 0 \Rightarrow e^{-i(H_1-H_2)t}|\psi_\beta\rangle = |\psi_\beta\rangle$
5) $H = \hbar\omega(a^\dagger a + \frac{1}{2})$ for harmonic oscillator

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$$

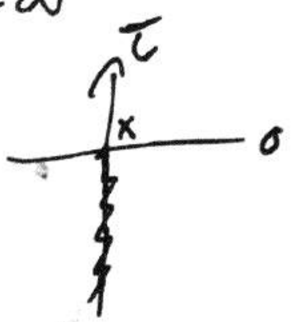
$$\Rightarrow |\psi_\beta\rangle = \frac{e^{-\frac{1}{4}\beta\hbar\omega}}{\sqrt{Z_\beta}} e^{-\frac{1}{2}\beta\hbar\omega a_1^\dagger a_2^\dagger} |0\rangle_1 \otimes |0\rangle_2$$

Recall: Path integral for vacuum state

$\psi(x) = \langle x | 0 \rangle$ $\qquad t \to -i\tau$

$$= c \int_{\substack{x(\tau_0)=0 \\ \tau_0 \to -\infty}}^{x(\tau=0)=x} DX(\tau)\, e^{-S_E[x(\tau)]/\hbar}$$

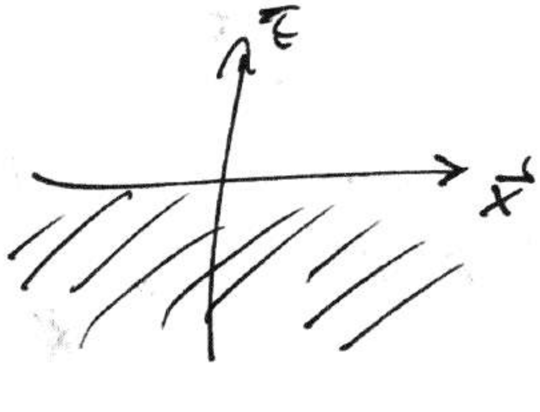
$$= \lim_{\tau \to -\infty} \langle x | e^{\tau H} | 0 \rangle$$

$=$ 

$ds^2 = -dt^2 + d\vec{x}^2$

$\downarrow$

$ds_E^2 = d\tau^2 + d\vec{x}^2$



$$\Psi_0[\varphi(\vec{x})] = \langle \varphi(\vec{x}) | 0 \rangle$$

$$= c \int_{\varphi(\tau \to -\infty) \to 0}^{\varphi(\tau=0,\vec{x})=\varphi(\vec{x})} D\varphi\, e^{-S_E[\varphi]/\hbar}$$

$\langle x_2 | \tilde{n} \rangle_2 = {}_2\langle n | x_2 \rangle$

$= {}_1\langle n | x_2 \rangle$

$$\Psi_\beta(x_1, x_2) = \langle x_1\, x_2 | \Psi_\beta \rangle$$

$$= \frac{1}{\sqrt{Z_\beta}} \sum_n e^{-\frac{1}{2}\beta E_n} \langle x_1 | n \rangle_1 \langle x_2 | \tilde{n} \rangle_2$$

$$= \sum_n e^{-\frac{1}{2}\beta E_n} \langle x_1 | n \rangle \langle n | x_2 \rangle$$

$$= \langle x_1 | e^{-\frac{1}{2}\beta H} | x_2 \rangle$$

$$= \langle x_1 | e^{-\frac{i H}{\hbar}\Delta t} | x_2 \rangle \Big|_{\Delta t = -\frac{i}{2}\hbar\beta}$$

$$\rightsquigarrow = \frac{1}{\sqrt{Z_\beta}} \int_{x(-\frac{1}{2}\beta\hbar)=x_2}^{x(0)=x_1} Dx(\tau)\, e^{-\frac{1}{\hbar} S_E[x(\tau)]}$$

$$= \int_{x_2,\ -\frac{1}{2}\beta\hbar}^{x_1,\ 0} =$$

$$\Rightarrow \langle \psi_\beta | \psi_\beta \rangle = \frac{1}{Z_\beta} \times \int_{x(-\frac{1}{2}\beta\hbar)=x_2}^{x(0)=x_1} Dx(\tau) \int_{\tilde{x}(\frac{1}{2}\beta\hbar)=x_2}^{\tilde{x}(0)=x_1} D\tilde{x}(\tau)\, e^{-S_E[x]}$$

assume S_E invariant under $\tau \to -\tau$

$$= \frac{1}{Z_\beta} \int_{x(-\frac{1}{2}\beta\hbar)=x(\frac{1}{2}\beta\hbar)} Dx(\tau)\, e^{-S_E[x]} = 1$$

$=$

$=$

Field theory:

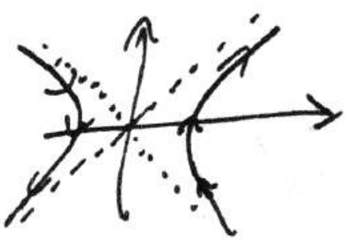
$$\langle \varphi_1(\vec{x})\, \varphi_2(\vec{x}) | \Psi_\beta \rangle$$

$$= \frac{1}{\sqrt{Z_\beta}} \int_{\varphi(-\frac{1}{2}\beta\hbar, \vec{x}) = \varphi_2(\vec{x})}^{\varphi(0,\vec{x}) = \varphi_1(\vec{x})} D\varphi(\tau, x)\, e^{-S_E[\varphi]} \qquad (*)$$

B. Unruh temperature from entanglement

$$ds^2 = -dT^2 + dX^2 \qquad X = \rho \cosh\eta \quad \swarrow R \text{ patch}$$

$$= -\rho^2 d\eta^2 + d\rho^2 \qquad T = \rho \sinh\eta$$



$\eta \to \eta + \text{const} \Rightarrow$ boost in (T, X)

similarly:

$$X = -\rho \cosh\eta \quad \swarrow L \text{ patch}$$

$$T = -\rho \sinh\eta$$

R, L are causally disconnected

Three sets of observers:

Mink: "see" the entire Mink space

Rind_R: ...... R patch

Rind_L: ...... L patch

Mink : Cauchy slice : $T=0$

$\mathcal{H}_{Mink}$: span$\{|\varphi(\vec{x})\rangle\}$

$\varphi(X) = \varphi(T=0, X)$

H_M : using T as time

$|0\rangle_M$

Rind$_R$: Cauchy slice $\eta = 0$ ($X > 0$ axis)

$\mathcal{H}_{Rind_R}$: span$\{|\varphi_R(\rho)\rangle\}$

$\varphi_R(\rho) = \varphi(T=0, X=\rho>0)$

H_R : obtained from S restricted to R with η as time

$|0\rangle_R$

Rind$_L$: Cauchy slice : $\eta = 0$ ($X<0$ axis)

$\mathcal{H}_{Rind_L}$: span$\{|\varphi_L(\rho)\rangle\}$

$\varphi_L(\rho) = \varphi(T=0, X=-\rho<0)$

Since $\Phi(\underline{X}) = (\varphi_L(\rho), \varphi_R(\rho))$

$$|\varphi(\underline{X})\rangle = \underbrace{|\varphi_L(\rho)\rangle \otimes |\varphi_R(\rho)\rangle}_{\sim PT}$$

$$\Rightarrow \mathcal{H}_{MinK} = \mathcal{H}_{Rind_L} \otimes \mathcal{H}_{Rind_R}$$

Question:
is $|0\rangle_M$ equivalent to $|0\rangle_L \otimes |0\rangle_R$?

Answer:
It turns out, no.

Note: any field theory is CPT-invariant.

$$\Rightarrow \mathcal{H}_R \overset{\sim}{\leftrightarrow} \mathcal{H}_L$$

(R, L) form a double

Claim: $|0\rangle_M$ is a TFD for $\mathcal{H}_{Rind_L} \otimes \mathcal{H}_{Rind_R}$

strategy for proof: coordinate space ~~wave~~ wavefunction

Note: (T_E, X)-LHP in fact coincides with Euclidean analytic continuation of Rindler

$$\eta \to -i\theta$$
$$\theta \in (-\pi, 0)$$

$$\Rightarrow \Psi_0[\varphi(x)] = \int_{\varphi(\theta=-\pi,\rho)=\varphi_L(\rho)}^{\varphi(\theta=0,\rho)=\varphi_R(\rho)} D\varphi(\theta,\rho)\, e^{-S_E[\varphi]} \qquad (**)$$

compare $(*)$ w/ $(**)$

$$\Rightarrow \Psi_0[\varphi(x)] = \langle \varphi_R(\rho)\, \varphi_L(\rho) \,|\, \Psi_\beta \rangle$$

with $\frac{1}{2}\beta\hbar = \pi$

$$\Rightarrow \beta = \frac{2\pi}{\hbar}$$

$$\Rightarrow Z_0^{(Mink)} = Z_{\beta=\frac{2\pi}{\hbar}}^{(Rind)}$$

We conclude:

$$|0\rangle_M = |\Psi_{\beta=\frac{2\pi}{\hbar}}\rangle$$

$$Z_0 = \mathrm{Tr}\left(e^{-\frac{2\pi}{\hbar} H_{Rind}}\right)$$

since β is associated with η

$$dt_{loc} = \rho\, d\eta$$

$$\Rightarrow \boxed{\beta_{loc} = \frac{2\pi\rho}{\hbar}}$$

just as we derived last time but this time from real-time wavefunction

Sept 24 (Missed, got notes from Sam)

Remarks

1) Euclidean method: regularity of analytic continuation
$\Rightarrow$ only have equilibrium at T_U

Now: When system is at $|0\rangle_M$
$\Rightarrow$ R/L observers both thermal at T_U

$$\underset{\text{Mink}}{\underset{\uparrow}{Z_0}} = \underset{\text{Rind}}{\underset{\uparrow}{Z_{\beta = 2\pi/\hbar}}}$$

2) Thermal nature comes from
(a) Special entangled structure of $|0\rangle_M$
(b) Tracing out the other half

3) Both derivations used a simple geometric feature:
Euclidean analytic cont. of Mink_2
$\shortparallel$
Euclidean analytic cont. of Rind
+ special periodicity

$\rightsquigarrow$ This is very general, applies to any QFT

4) Entanglement method: no need to deal w/ conical singularity

5) $(H_L - H_R)|\Psi_\beta\rangle = 0 \rightsquigarrow e^{-i\eta(H_L - H_R)}|\Psi_\beta\rangle = |\Psi_\beta\rangle$

Boost inv. of vacuum $\rightsquigarrow e^{-i\eta(H_L - H_R)}|0\rangle_M = |0\rangle_M$

$H_R = \int_0^\infty dX\, X\, T_{00}$ (on $T=0$ Cauchy slice)

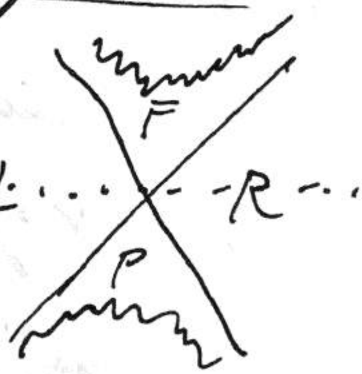
$H_L = \int_{-\infty}^0 dX\,(-X)\,T_{00}$ (on ")

C: Hawking Temperature from entanglement

$$ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2$$

$$= g(r)\left(-dT^2 + dX^2\right) + r^2 d\Omega^2$$

$$X^2 - T^2 = \frac{1}{\kappa^2} e^{\frac{r-r_s}{r_s}}\left(\frac{r-r_s}{r_s}\right)$$



Similarity: Kruskal observers $\to T$

Schwarzschild observers $\to t$

$$\mathcal{H}_K = \mathcal{H}_L \otimes \mathcal{H}_R$$

Important difference from Rindler

→ Metric is not T-independent ⇒ H_K is not either ⇒ energy not conserved

no notion of vacuum state

Nevertheless we can define the counterpart of $|0\rangle_M$ using path integrals to get $|0\rangle_{HH}$ "Hartle-Hawking vacuum"

Key:
(1) Kruskal metric allows a sensible $T \to -iT_E$
(2) Euclidean manifold is again the same as that obtained by taking $t \to i\tau$ with $\tau \sim \tau + \frac{2\pi}{\kappa}$

$\Rightarrow$ (def) $|0\rangle_{HH} :=$ path integral over $T_E < 0$

$$|0\rangle_{HH} = |\Psi_{\beta_H}\rangle \leftarrow \text{Thermal field double with } \beta_H = \frac{2\pi}{\hbar\kappa}$$

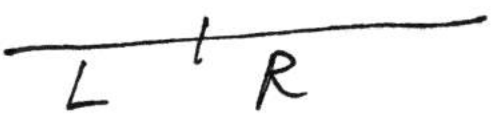
D: Geometry & Entanglement

Previously: From perspective of Rindler or Schwarzschild observers, there is a singular behaviour at the horizon unless they are at T_H

$\to$ explain this using entanglement

Rindler: $\eta \to -i\theta$, zero temp $\to$ θ not compact $\to$ $|0\rangle_R \otimes |0\rangle_L$ (*)
in this state L and R are unentangled

For any smooth wavefunction of any QFT in Mink_2



$\to$ always entangled for any finite energy state

For L, R not entangled, we'd need a barrier at $X=0$
$\Rightarrow$ singular behavior at $X=0$ that causally propagates

Note (*) is not a state in $Mink_2$

$|L\rangle \otimes \langle R| \leftrightarrow$ No FP regions

$\Rightarrow$ Presence of F/P regions in $Mink_2$

$\Updownarrow$

Entanglement of L, R.

Generalize:

1) Any finite energy state in $Mink_2$ with all 4 regions requires specific entanglement between L and R

2) Generic state in $\mathcal{H}_L \otimes \mathcal{H}_R$ doesn't have that entanglement structure

$\Rightarrow$ Cannot be interpretted as sensible in $Mink_2$

3) Similarly, generic state in $\mathcal{H}_L \otimes \mathcal{H}_R$ for schwarzschild will have "fire wall" at the horizon

4) discussion is at the level of states, independent of details about $\mathcal{H}_M$, $\mathcal{H}_L$, $\mathcal{H}_R$, etc.

1.3.3 Free Field Theories Derivation

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_n e^{-\frac{1}{2}\beta E_n} |\tilde{n}\rangle_L \otimes |n\rangle_R = \frac{e^{-\frac{1}{4}\beta\hbar\omega}}{\sqrt{Z_\beta}} e^{e^{-\frac{1}{2}\beta\hbar\omega} a_1^\dagger a_2^\dagger} |0\rangle_1 \otimes |0\rangle_2$$

Note $\quad a\, e^{\alpha a^\dagger} |0\rangle = \alpha\, e^{\alpha a^\dagger} |0\rangle$ $\qquad a_1 |0\rangle_1 = 0$, $\quad a_2 |0\rangle_2 = 0$

$$\Rightarrow \left(a_1 - e^{-\frac{1}{2}\beta\hbar\omega} a_2^\dagger\right) |\Psi_\beta\rangle = 0$$

$$\left(a_2 - e^{-\frac{1}{2}\beta\hbar\omega} a_1^\dagger\right) |\Psi_\beta\rangle = 0$$

def: $\quad b_1 = \cosh\theta\, a_1 - \sinh\theta\, a_2^\dagger$

$\qquad b_2 = \cosh\theta\, a_2 - \sinh\theta\, a_1^\dagger$

$\Rightarrow$ take $\cosh\theta = \dfrac{1}{\sqrt{1 - e^{-\beta\hbar\omega}}}$, $\sinh\theta = \dfrac{e^{-\frac{1}{2}\beta\hbar\omega}}{\sqrt{1 - e^{-\beta\hbar\omega}}}$

then $[b_1, b_1^\dagger] = [b_2, b_2^\dagger] = 1$

else $[\cdot, \cdot] = 0$

and $b_1 |\Psi_\beta\rangle = b_2 |\Psi_\beta\rangle = 0$

$\Rightarrow |\Psi_\beta\rangle$ is vacuum for b_1, b_2

Free Field theories $\rightarrow$ a bunch of harmonic oscillators

"$|0\rangle_1, |0\rangle_2$" $\rightarrow$ "$|0\rangle_L\ |0\rangle_R$", "$|\Psi_\beta\rangle$" $\rightarrow$ "$|0\rangle_M$"

Free Massless scalar:

$$S = -\frac{1}{2}\int d^2x\, \partial^\mu \varphi\, \partial_\mu \varphi$$

Minkowski observers

$(-\partial_T^2 + \partial_X^2)\varphi = 0$

$\Rightarrow u_p = \frac{1}{\sqrt{2\omega_p}} e^{-i\omega_p T + ipX}$

$\omega_p = |p|$

$U = T - X$, $V = T + X$ $\Rightarrow u_p = \begin{cases} \frac{1}{\sqrt{2\omega_p}} e^{-i\omega_p U}, & p>0 \\ \frac{1}{\sqrt{2\omega_p}} e^{-i\omega_p V}, & p<0 \end{cases}$

$(\varphi_2, \varphi_1) := i\int_{-\infty}^{\infty} \left(\varphi_2^* \partial_T \varphi_1 - (\partial_T \varphi_2^*)\varphi_1\right)$

$$\underbrace{\begin{aligned}(u_p, u_{p'}) &= 2\pi\,\delta(p-p') \\ (u_p^*, u_{p'}^*) &= -2\pi\,\delta(p-p') \\ (u_p, u_{p'}^*) &= 0\end{aligned}}$$

CCR with $\varphi = \sum_p (a_p u_p + a_p^\dagger u_p^*)$

$[a_p, a_{p'}^\dagger] = 2\pi\,\delta(p-p')$

$a_p |0\rangle_M = 0$

Rindler "R"-observers

$ds^2 = \underbrace{-dT^2 + dX^2}_{\text{Mink}} = -\rho^2 d\eta^2 + d\rho^2$

$\rho = e^{\xi} \Rightarrow ds^2 = e^{2\xi}\overbrace{(-d\eta^2 + d\xi^2)}^{\text{Rind}}$

$(-\partial_\eta^2 + \partial_\xi^2)\varphi = 0$

$u = \eta - \xi$, $v = \eta + \xi$ $\Rightarrow v_k = \begin{cases} \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k u} \\ \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k v} \end{cases}$

$U = -e^{-u}$
$V = e^{v}$

$\varphi_R = \sum_k \left(b_k^{(R)} v_k + b_k^{(R)\dagger} v_k^*\right)$

$[b_k, b_{k'}^\dagger] = \underset{\text{metric?}}{2\pi}\,\delta(k-k')$

$b_k^{(R)} |0\rangle_R = 0$

Sept 26 (Missed, got notes from Sam)

Recall $U = -e^{-u}$, $V = e^{v}$ (So $U<0$, $V>0$)

$\Rightarrow$ Rindler "R" modes become

$$v_k = \begin{cases} \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k u} \\ \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k v} \end{cases} = \begin{cases} \frac{1}{\sqrt{2\omega_k}} e^{i\omega_k \log(-U)} & k>0 \\ \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k \log(V)} & k<0 \end{cases}$$

$$\varphi_R = \sum_k b_k^{(R)} v_k + b_k^{(R)\dagger} v_k^* \Rightarrow [b_k^{(R)}, b_{k'}^{(R)\dagger}] = 2\pi\delta(k-k')$$

Define right vacuum $\boxed{b_k^{(R)} |0\rangle_R = 0}$ Note $(\varphi_R^{(2)}, \varphi_R^{(1)}) = i\int_{-\infty}^{\infty} d\xi (\varphi_R^{(2)*} \partial_\eta \varphi_R^{(1)} - (\partial_\eta \varphi_R^{(2)})^* \varphi_R^{(1)})$

v_k is singular at $U=0$ and $V=0$ since the modes are only supported in the R region
$\Rightarrow$ potential singular behavior in physical quantities

Rindler "L":

$T = -e^{\xi} \sinh\eta$, $X = -e^{\xi}\cosh\eta$, $U = e^{-u}$, $V = -e^{v}$
$\quad >0 \qquad <0$

$$v_k = \begin{cases} \frac{1}{\sqrt{2\omega_k}} e^{i\omega_k u} \\ \frac{1}{\sqrt{2\omega_k}} e^{i\omega_k v} \end{cases} = \begin{cases} \frac{1}{\sqrt{2\omega_k}} e^{-i\omega \log(U)} & k>0 \\ \frac{1}{\sqrt{2\omega_k}} e^{i\omega \log(-V)} & k<0 \end{cases}$$

we have chosen positive-frequency modes to be the PT reversed of v_k

PT reversal ensures $k>0$ has u, $k<0$ has v

Note: $(\varphi_L^{(2)}, \varphi_L^{(1)}) = -i\int_{-\infty}^{\infty} d\xi (\varphi_L^{(2)*} \partial_\eta \varphi_L^{(1)} - (\partial_\eta \varphi_L^{(2)})^* \varphi_L^{(1)})$

(recall $|\Psi_\beta\rangle = \sum_n e^{-\frac{\beta E_n}{2}} |n\rangle_L |n\rangle_R$)

So $\varphi_L = \sum_k \left(b_k^{(L)} w_k + b_k^{(L)\dagger} w_k^* \right) \rightarrow [b_k^{(L)}, b_{k'}^{(L)\dagger}] = 2\pi\delta(k-k')$

Left vacuum $\boxed{b_k^{(L)} |0\rangle_L = 0}$

On Mink_2, $\Phi(T, X) = (\varphi_L, \varphi_R)$

$\Rightarrow \varphi = \sum_p \left(a_p u_p + a_p^\dagger u_p^* \right) = \sum_k \left(b_k^L w_k + b_k^{L\dagger} w_k^* + b_k^R v_k + b_k^{R\dagger} v_k^* \right)$

To find relation between $\{a_p, a_p^\dagger\}$ and $\{b_k^L, b_k^{L\dagger}, b_k^R, b_k^{R\dagger}\}$, need to find relations between $\{u_p, u_p^*\}$ and $\{w_k, w_k^*, v_k, v_k^*\}$

Two possibilities:

(1) $v_k = \sum_p c_{kp} u_p$, $w_k = \sum_p \tilde{c}_{kp} u_p$ without u^* involved

$\Rightarrow$ positive frequency modes of both L and R observers are related to only positive freq. Minkowski modes

$\Rightarrow b_k^{(R)} = \sum_p d_{pk} a_p$, $b_k^{(L)} = \sum_p \tilde{d}_{pk} a_p$

$\Rightarrow |0\rangle_M$ coincides with $|0\rangle_L \otimes |0\rangle_R$

(2) suppose $u_p = \sum_k \left(d_{pk} v_k + \tilde{d}_{pk} w_k + e_{pk} v_k^* + \tilde{e}_{pk} w_k^* \right)$

then $b_k^{(R)} = \sum_p \left(d_{pk} a_p + e_{pk}^* a_p^\dagger \right)$, $b_k^{(L)} = \sum_p \left(\tilde{d}_{pk} a_p + \tilde{e}_{pk}^* a_p^\dagger \right)$

Bogoliubov Transformations

$\Rightarrow \boxed{|0\rangle_M \neq |0\rangle_L \otimes |0\rangle_R}$

Requiring $b_k^{(R)} |0\rangle_L \otimes |0\rangle_R = b_k^{(L)} |0\rangle_L \otimes |0\rangle_R = 0$

we get: $|0\rangle_L \otimes |0\rangle_R \sim e^{\# a^\dagger a^\dagger} |0\rangle_M$

$$|0\rangle_M \sim e^{\# b^\dagger b^\dagger} |0\rangle_L \otimes |0\rangle_R$$

Focus on right-moving modes:

$$u_p = \frac{1}{\sqrt{2\omega_p}} e^{-i\omega_p U}, \quad v_p = \begin{cases} \frac{1}{\sqrt{2\omega_p}} e^{i\omega_p \log(-U)} & \text{for } U<0 \\ 0 & \text{for } U>0 \end{cases}, \quad w_k = \begin{cases} \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k \log U} & \text{for } U>0 \\ 0 & \text{for } U<0 \end{cases}$$

argument for possibility (2):

u_p is analytic in lower complex U plane

→ So is any linear superposition

Neither v_k or w_k is analytic there

⇒ v_k, w_k must involve both u_p, u_p^*

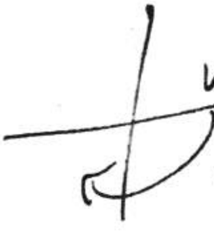
Instead of finding $d_{pk}, \tilde{d}_{pk}, e_{pk}, \tilde{e}_{pk}$ explicitly (which is doable)

consider another basis equivalent to (u_p, u_p^*)

(i.e. has the same vacuum) but related to

v, v^*, w, w^* in a simple way:

$\llcorner U$



analytic continuation in LH plane

construct λ_k from analytic cont. of v_k to $U>0$ in lower half plane

$$\Rightarrow \lambda_k = \frac{1}{\sqrt{2\sinh \pi\omega_k}} \left[e^{\frac{\pi}{2}\omega_k} v_k + e^{-\frac{\pi}{2}\omega_k} w_k^* \right]$$

$$\chi_k = \frac{1}{\sqrt{2\sinh \pi\omega_k}} \left[e^{\frac{\pi}{2}\omega_k} w_k + e^{-\frac{\pi}{2}\omega_k} v_k^* \right]$$

$\{\lambda_k, \chi_k\}$ share the same vacuum, $|0\rangle_M$ as $\{u_k\}$

$$\Rightarrow \varphi = \sum_k \left[c_k \lambda_k + d_k \chi_k + h.c.\right], \quad c_k|0\rangle_M = d_k|0\rangle_M = 0$$

$$c_k = \cosh\theta_k \, b_k^{(R)} - \sinh\theta_k \, b_k^{(L)\dagger}$$

$$d_k = \cosh\theta_k \, b_k^{(L)} - \sinh\theta_k \, b_k^{(R)\dagger}$$

where
$$\cosh\theta_k = \frac{e^{\frac{1}{2}\pi\omega_k}}{\sqrt{2\sinh\pi\omega_k}}$$
$$\sinh\theta_k = \frac{e^{-\frac{1}{2}\pi\omega_k}}{\sqrt{2\sinh\pi\omega_k}}$$

$$\Rightarrow |0\rangle_M = \prod_k \left(\frac{e^{-\frac{\pi}{2}\omega_k}}{\sqrt{Z_k}}\right) \exp\left[\sum_k e^{-\pi\omega_k} b_k^{(R)\dagger} b_k^{(L)\dagger}\right] |0\rangle_L \otimes |0\rangle_R$$

with $Z_k = \dfrac{1}{2\sinh\pi\omega_k}$ $\Big\}$ for each k this is exactly the result for a QHO at $\beta = \frac{2\pi}{\hbar}$

(Free) <u>Massive scalar in Schwarzschild background</u>:

$|0\rangle_{HH} \leftarrow$ squeezed state for Schwarzschild obs.

- Realistic BH

$\leftarrow$ get temp. from free field theory

1.4 BH Thermodynamics

BH has a temperature

$$T_H = \frac{\hbar \kappa}{2\pi} = \frac{\hbar}{8\pi G_N M} \quad (1)$$

$\Rightarrow$ natural to interpret it as a thermodynamic system

suppose it has an entropy S.

Expect it to satisfy 1st law:

$$dE = T\,dS \quad (2)$$

identify $E = M$, integrate (2) to find S

$$dS = \frac{dM}{T} \Rightarrow \boxed{S = \frac{4\pi G_N M^2}{\hbar}}$$

but $r_s = 2G_N M$

$$\Rightarrow \boxed{S = \frac{4\pi r_s^2}{G_N \hbar} = \frac{A}{4\hbar G_N}} \quad (3)$$

So using (1) and (3), we can rewrite (2) as

$$dM = \frac{\hbar \kappa}{2\pi} \frac{A}{4\hbar G_N} = \frac{\kappa}{8\pi G_N} dA \quad (4)$$

(4) is a pure geometric relation

Eq. (4) is part of a set of four laws on general BHs called "Four laws of BH mechanics"

• No hair theorem:

A stationary asymptotically flat BH is solely characterized by:

1) mass M
2) angular momentum J
3) electric or magnetic charges Q

• Four laws: (1972)

0th law: Surface gravity κ is constant over the horizon

1st law: $dM = \frac{\kappa}{8\pi G_N} dA + \Omega\, dJ + \Phi\, dQ$

Ω: angular frequency at the horizon
φ: electric potential (s.t. $\varphi(\infty)=0$)

2nd law: Horizon area never decreases classically.

3rd law: Surface gravity of a BH cannot be reduced to 0 in a finite sequence of operations

With (1) + (3), the four laws of BH mechanics become the standard laws of thermodynamics

Beckenstein (1972-1974):

BH should have an entropy $\propto A$

otherwise the second law of thermodynamics would be violated in the presence of a BH

Define

$$S_{tot} := S_{matter} + S_{BH}$$

$\rightsquigarrow$ Generalized second law

$\underline{\Delta S_{tot} \geq 0}$

with (1975) Hawking radiation, GSL becomes standard 2nd law.

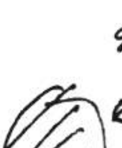
Remarks:

1) in classical limit $\hbar \to 0$ (G_N fixed)

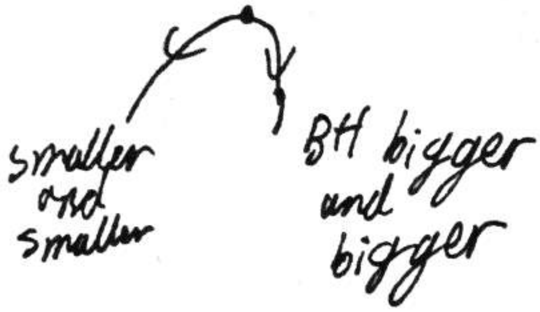
$$T_H \to 0, \quad S_{BH} \to \infty$$

2) $T_H \propto \frac{1}{M}$, $M\uparrow \Rightarrow T_H \downarrow$

$$\Rightarrow \underset{\substack{\uparrow \\ \text{specific heat}}}{C} = \frac{\partial M}{\partial T} = -\frac{\hbar}{8\pi G_N} \frac{1}{T^2} < 0 \Rightarrow \text{negative specific heat}$$

$\Rightarrow$  $\leftarrow T_H$ not a stable equilibrium. Why? Consider small fluctuations in T_{BH}, say $T_{BH}\uparrow$, radiate a bit more to env $\Rightarrow M\downarrow \Rightarrow T_{BH}\uparrow \Rightarrow$ radiate <u>even</u> <u>more</u> and similarly unstable in the other direction.

BH + infinite bath



Stable equilibrium is possible in a finite box

3) $T_H = \frac{\hbar \kappa}{2\pi}$ and $S = \frac{A}{4\hbar G_N}$

. Universal

Apply to any matter coupled to Einstein gravity! (AdS, dS, Mink, all spacetimes)

4) With higher derivative corrections to Einstein gravity
(i.e. $R^2 + \lambda^2 R_{\mu\nu}^2 + R^2 + \cdots$)
these ~~rules~~ equations no longer apply but S, T_H can still be expressed in terms of horizon quantities

1.5 Quantum Nature of Black Holes and the Holographic Principle

BH thermodynamics + $T_H \propto \hbar$
$S_{BH} \propto 1/\hbar$

$\Rightarrow$ Natural to treat BH as a <u>macroscopic quantum statistical system.</u>

Questions:

(1) What is the statistical interpretation of the entropy of a black hole?

From standard stat Mech:

$\#$ microstates $= \Omega$ (consistent w/ a given macroscopic equilibrium)

$\Rightarrow \quad S = k_B \log \Omega$

For BH, expect $\Omega = e^{\frac{A}{4\hbar G_N}} = e^{\frac{A}{4 l_p^2}}$

Heuristically:

entropy ~ put $\underbrace{1 \text{ d.o.f.}}_{\text{(e.g. spin)}}$ in each Planckian cell.

(a) This is a huge entropy

For $M_{BH} = M_{\odot}$, $r_S = 3\,\mathrm{km} \Rightarrow \frac{A}{4\hbar G_N} \simeq 1.1 \times 10^{77}$

$\Rightarrow \Omega \approx e^{10^{77}}$

The sun itself has entropy $\frac{S}{\hbar} \approx 10^{57}$

(b) When a star collapses to form a BH, there is a huge increase in # of available microstates

no hair theorem: all these states must be quantum mechanical in nature.

huge increase $\Leftrightarrow$ gravity is weak (Planck scale is very small)

(c) In string theory, there are black holes whose microstates can be precisely counted. giving (after complicated combinatorics), exactly an entropy of $S = \frac{A}{4\hbar G_N}$

(d) In holographic duality, for black holes in AdS, the statistical origin is again known

(2) Hawking's information loss paradox

Hawking:

1) To an excellent approximation BH radiates thermally for $M \gg M_{pl}$
"white noise"

2) BH loses mass

3) should disappear

But when $M \sim M_{pl}$, not enough d.o.f. to encode all the information put into it

Another way to say this:

Suppose a star is in a pure state

$\Rightarrow$ BH

$\Rightarrow$ Radiates

$\Rightarrow$ Radiation (mixed (thermal) density matrix)

$\Rightarrow$ 3 logical possibilities:

1) Information is lost $\Rightarrow$ QM must be modified

2) Hawking radiation stops at $M \sim O(M_{pl})$ R $\otimes$ Radiation

$\Rightarrow$ Planckian mass remnant is left, which encodes all information

3) No remnants, unitary evolution $\Rightarrow$ information comes out from radiation

1) Is the most radical. It is also fiendishly difficult to modify QM

2) Blames unknowns ⟸ another universe

A variant: 

3) Is the most conservative

⇝ significant challenges still to explain <u>how</u> the information comes out from radiation

⇒ imply quantum gravity puts highly nontrivial constraints/implications on low-energy physics

simpler question: Burning of coal

- Preparation: A <u>typical</u> <u>highly excited</u> <u>pure state</u> in a <u>non-integrable</u> <u>many-body</u> system <u>looks</u> thermal if one only probes a small part of it.

Say I separate the system in two:

$$A + B$$

$$\rho_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$$ is very close to being thermal

"trace distance is exponentially [illegible] in [illegible] to therm[al]"

provided $|A| \lesssim |B|$

3 cont. ⇒ One reveals a given state is pure only by having full global information

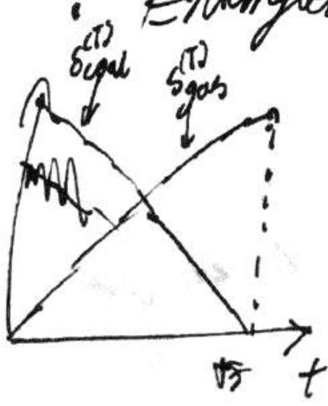
some remarks:

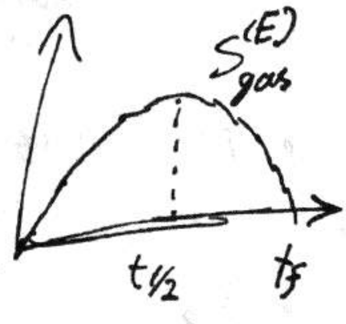
(a) At a given time, emitted photons look almost perfectly thermal.

(b) Nevertheless, they do contain information, but in a very subtle way.
The information is encoded in the entanglement with the rest of the system. → $e^{-N} \sim e^{1/G_N}$ ← nonperturbative

(c) Consider the following quantities:

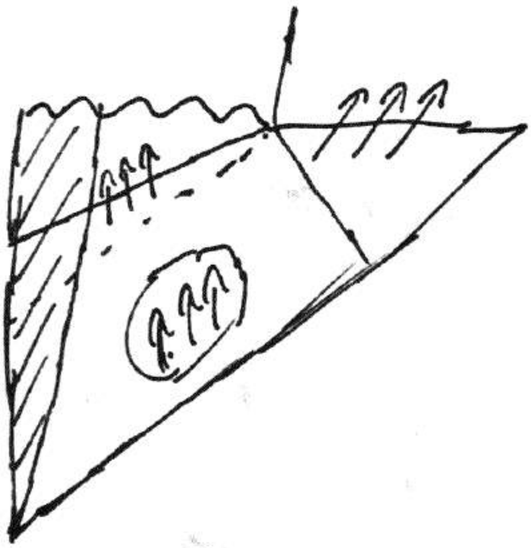
- Thermal entropy of photon gas: $S^{(T)}_{gas}$
- Thermal entropy of the coal: $S^{(T)}_{coal}$ } "coarse-grained"
- Entanglement entropy of photon gas: $S^{(E)}_{gas}$
- Entanglement entropy of the coal: $S^{(E)}_{coal}$ } "fine-grained"





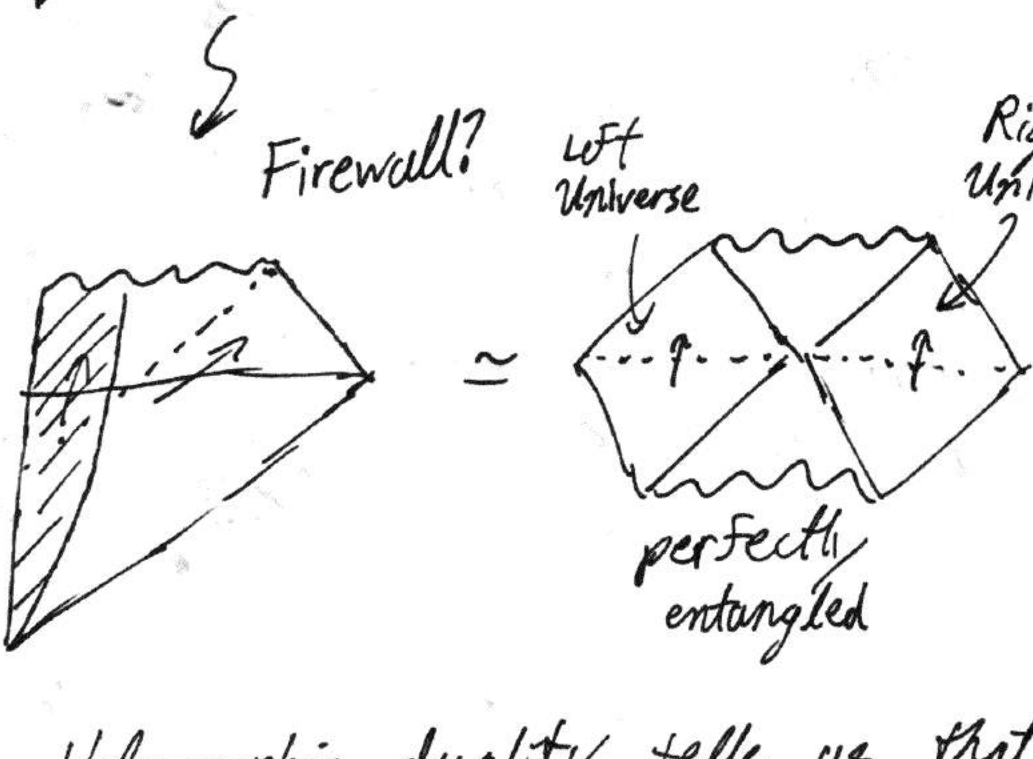
$S(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A)$

This is a good paradigm for BH evaporation but BH is not coal.
Coal is causally connected with emitted radiation.
Black hole's infalling matter is causally disconnected with the emitted radiation.



Would either violate

- No-cloning theorem (QM)
- Locality (QFT)



Holographic duality tells us that information evaporation for a Black Hole should be just like the burning of coal

· Entropy bounds and holographic principle

starting point:
BH is a quantum statistical system
+ a couple of "facts"
⇒ entropy bounds and holography

Facts:

(1) A sufficiently massive object in a compact volume always collapses to form a Black Hole

Rule of thumb: if $2G_N M > L \Rightarrow BH$

(2) Entropy reflects # of degrees of freedom

$$\rho \to S = -\mathrm{Tr}[\rho \log \rho]$$

⇒ for a system with N-dimensional Hilbert space

$$S_{max} = \log N$$

(a) for a system of spins $N = 2^n \Rightarrow S_{max} \sim n$

(b) $\mathcal{H}$ for h.o. is infinite-dimensional, but for finite energy, $\mathcal{H}$ is f.d.

* Spherical entropy bound

⇒ Take an isolated system of energy E, entropy S_0 in asymptotically flat spacetime

Let A be the area of smallest sphere enclosing the system, M_A be the mass of a black hole with that area.

Then $E \leq M_A$

$\Rightarrow$ Maximal ~~entropy~~ energy one could add (keeping A fixed) is

$M_A - E$

$S_{final} = S_{BH} \geq S_{init}$
$= S_0 + S'$

$$S_0 \leq S_{BH} = \frac{A}{4\hbar G_N} \Rightarrow S_{max} = \frac{A}{4 l_p^2}$$

Remarks:

1) S is ~~classically~~ extensive $\propto V$
$\Rightarrow$ QG behaves very differently from non-gravitational systems

2) A cubic lattice of spins (size L, spacing a) has $S_{max} = \frac{L^3}{a^3} \log 2$ (*)

At $G_N = 0$

Now slowly increase G_N (i.e. l_p)

Then (*) violates the bound when

$$\frac{L^3}{a^3} \log 2 \gtrsim \frac{A}{4 l_p^2} \qquad A \sim L^2$$

$$\frac{l_p^2}{L^2} \gtrsim \frac{a}{L} \cdot \frac{a^2}{L^2}$$

Suppose each site has mass m

$$M = \frac{L^3}{a^3} m$$

total system is a Black hole when

$$\frac{\hbar \overbrace{G_N}^{l_p^2} M}{\hbar} > L \Rightarrow l_p^2 \frac{L^3}{a^3} \frac{m}{\hbar/c} > L$$

$$\Rightarrow \frac{l_p^2}{a^2} > \frac{a}{L}\frac{l_c}{L} = \lambda_2$$

$$\lambda_2 \ll \lambda_1$$

So (a) $S \propto A$
and (b) $S \propto V$
But <u>both</u> conditions are important:

a) In a closed universe S^3

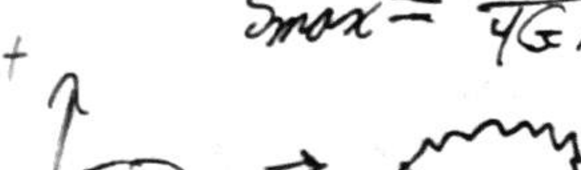
$$A = 0, \quad S \neq 0$$

trivially violated

b) consider non-spherical region in asymptotically flat universe
take V space-like, with boundary ∂V

$$S_{max} \stackrel{?}{=} \frac{A(\partial V)}{4 G \hbar}$$

(a)+(b): For a general region, V, with bdy ∂V, in general ∂V has not much to do with physics inside V

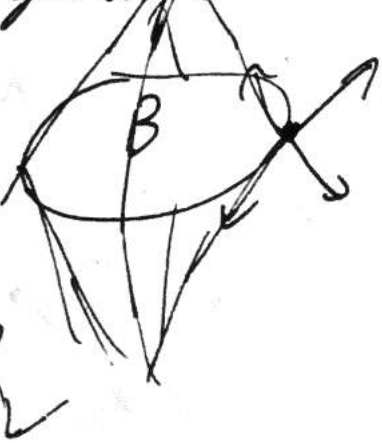


light-like $\Rightarrow A(\partial V) \approx 0$

I don't understand this

A generalization for asymptotic flat spacetimes (universes)

- Consider a general codimension 1 <u>spacelike</u> closed surface



$\Rightarrow$ 4 light rays

Future in / out

past in / out

causal diamond of B

c.f. hep-th/0203101

Take D to denote the causal diamond

- any point in D is fully determined by the information enclosed in B

- conjecture:

$$\text{Entropy on any Cauchy slice of } D \leq \frac{A_B}{4\hbar G_N}$$

This doesn't work in cosmological settings or inside a black hole.

Most general formulation:

For any B, construct light-sheet from B: null hypersurface formed by non-expanding light rays from B.

$$\Rightarrow \underbrace{S[L(B)]}_{\text{entropy of "matter" d.o.f. passing through light sheet.}} \leq \frac{A_B}{4\hbar G_N}$$

Entropy bound + entropy associated with # d.o.f.

$\Rightarrow$ statement on # of d.o.f.

$\Rightarrow$ Holographic principle

"When something is so new, that you don't have the language to describe it, you describe it in whatever language you can. It may not be precise, it may not even be true, but it's better than nothing."

~ Xiao-gang Wen

$\rightsquigarrow$ A spherical region of boundary area A can be fully described by no more than

$$\frac{A}{4\hbar G_N} = \frac{A}{4 l_p^2} \text{ d.o.f.}$$

i.e. ~ one degree of freedom per planck-area