Chapter 1: Black Holes & the Holographic Principle

BHs: 1) Key object - astrophysically ubiquitous

2) Quantum Matter around BH
   - Hawking's 1975 paper
     - Black Hole radiation
   
   ▶ Black holes bring quantum gravity to a macroscopic level.

1.1 General Remarks on Gravity

   all other interactions: probed to $10^{-33}$ cm (LHC)
   gravity: only $10^{-2}$ cm

   General Relativity: "gravity = spacetime"

   Quantum Gravity: " = quantum spacetime"
Question: What is the relationship between quantum gravitational effects and the nature of spacetime?

Answer: A) Einstein Gravity & Gravitons

Line element: \( ds^2 = g_{\mu\nu} dx^\mu dx^\nu \)

Einstein's Equations: \( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R - 2\Lambda) = 8\pi G_n T_{\mu\nu} \)

Factor: \( \Lambda \) is the cosmological constant. Matter (or stress-energy tensor).

Action: \( S = \frac{1}{16\pi G_n} \int d^4 x \sqrt{g} (R - 2\Lambda) + S_{\text{matter}} \)

\( T_{\mu\nu} = -\frac{1}{2} \frac{\partial}{\partial g_{\mu\nu}} S_{\text{matter}} \frac{\partial S_{\text{matter}}}{\partial g^{\mu\nu}} \)

Take \( \Lambda = 0, \quad T_{\mu\nu} = 0 \)

Simplest solution to (*) \( \rightarrow g_{\mu\nu}(\ast \ast) \)

"Weak gravity" \( g_{\mu\nu} = \eta_{\mu\nu} + x h_{\mu\nu} \) with \( x^2 \sim 8\pi G_n \)

Putting (1) into (\( \ast \ast \)), we get

\( S = \int d^4 x \left[ L_2 + x L_3 + x^2 L_4 + \ldots \right] + \frac{x h_{\mu\nu} T_{\mu\nu} + 1}{2} \)

Starts at \( O(x^2) \sim 0(1^2) \), cancels some terms.

Canonically normalized since two masses cancel.
EOM from $L_2$ give plane wave solutions

These are what we call "gravitational waves"

$L_2$ is quadratic in $h$ \iff free field theory $\forall h$

$\Rightarrow$ quantize $L_2 \Rightarrow$ spin-2 massless particle "graviton"

$\Rightarrow L_3, L_4, \ldots \Rightarrow$ self-interaction of the graviton

gravitational interaction of matter

is by exchange of $h$

e.g. electron as matter has $Tuv \sim \Phi\Phi$

$\Rightarrow \Box \Phi + \Box \Phi + \Box \Phi + \ldots$

Treated as QFT, $S$ is non-renormalizable.

$\Rightarrow$ so we can treat $S[\Phi]$ as an effective field theory

not fundamental.

B) Important scales of Gravity

Planck Planck scales: $\hbar, G_N, c = 1 \Rightarrow$

$M_p = \sqrt{\frac{\hbar}{G_N}} \approx 1.2 \times 10^{19}$ GeV

$\ell_p = \frac{\hbar}{2M_p} = \sqrt{\frac{\hbar}{G_N}} = 1.6 \times 10^{-33}$ cm

$t_p = \ell_p c \approx 5.4 \times 10^{-22}$ sec

For reference: top quark mass $\approx 10^{-17} M_p$

electron mass $\approx 10^{-23} M_p$

Largest of $M_p \ll$ weakness of gravity at microscopic scales

Consider two particles of mass $m$ brought to nearest possible distance

$\Rightarrow x_c = \frac{V_0(r_c)}{m} = \frac{G_N m^2}{r_c} = \frac{G_N m^2}{M_p^2} \approx \frac{m^2}{M_p}$

$r_c = \frac{m}{M_p} \ 	ext{Compton wavelength}$
We see $\lambda_\xi = \frac{p^2}{M_p^2} = \frac{\lambda^2}{r_c^2}$. For $m \ll M_p$, $\lambda_\xi \ll 1$ e.g $e^\frac{\lambda^2}{r_c^2}$

Relativistic calculation: $\lambda_\xi \sim \frac{E^2}{M_p^2}$, E c.o.m. energy

If we take $m \gg M_p$, does $\lambda_\xi \gg 1$? No!

Different question: take point particle of mass $m$.

At what distance $r_s$ from it does classical gravity become strong?

probe $m \rightarrow \text{c.o.m.}$ 

\[ \frac{G m m'}{r_s} \sim 1 \Rightarrow r_s \approx \frac{G N m}{m} \]

Remarks: 1) In Newtonian gravity, at $r_s$ escape velocity $\sim c$

2) In GR, $r_s \sim$ Schwarzschild radius of Black Hole

$\Rightarrow r_s \sim$ minimal distance we can probe an obj. in classical gravity.

Two important scales: $r_c \sim \frac{M_p}{m}$ $\Rightarrow \frac{r_s}{r_c} = \frac{G N m^2}{\frac{M_p}{m}} = \left(\frac{m}{M_p}\right)^2$

1) $m < M_p \Rightarrow r_s \ll r_c$ so Compton wavelength outside $r_s$ $\Rightarrow$ gravitation is weak, negligible $r_s$ not important quantum effects dominate

2) $m \sim M_p$, $r_s \sim r_c$, $\lambda_\xi \sim 1$ Quantum Gravity becomes important

3) $m > M_p$, $r_s > r_c$ not relevant $\Rightarrow$ classical gravity dominates

The relationship between Black Holes and quantum gravity, however, affects much more than Planck scale physics.
Last time:

\[ x_G \sim \frac{G_N E^2}{k} \sim \frac{E^2}{M_p^2} \]

where \[ k = 8\pi G_N \]

\[ k \ll \frac{1}{L^2} \] for \[ L \gg M_p \]

\[ \frac{r_s}{r_c} \sim \frac{m^2}{M_p^2} \]

Corollary: \[ L_p \] is minimal localization length

Non-grav: \[ S_R \sim T \] with gravity: \[ E \sim M_p \]

\[ r_s \sim \frac{G_N E}{r_c} \sim L_p \]

\[ E \gg M_p \]

\[ r_s \gg r_c \]

\[ S_R \sim \frac{\pi}{8L} \implies \sqrt{\frac{G_N}{S_X}} \sim L_p \]

\[ E \ll M_p \] ignore: (1) grav. interaction

(2) fluctuations 

\[ \sim \text{rigid spacetime} \quad \text{(can be curved)} \]

\[ \text{i.e. on earth} \]
Mathematical Treatment:

$E$ fixed, $\kappa$ fixed, $C_\nu \to 0$

($\nu = 0$, $\nu + \infty$)

C low energy expansion in $C_\nu$

$Z = \int Dg \; D\nu \; e^{iS[g,\nu]}$

$S = \frac{1}{16\pi C_\nu} \; S_{\text{grav}}[g] + \frac{1}{\lambda} \; S_{\text{m}}[g,\nu]$

$\lambda$: matter coupling

$\lambda \gg C_\nu$

$C_\nu \to 0 \Rightarrow$ saddle point: $SS_{\text{grav}}[g] = 0$

$\Rightarrow g_{\text{classical}}$

Expand $g = g_{\text{classical}} + \chi h$

$\Rightarrow S = \frac{1}{16\pi C_\nu} \; S_{\text{grav}}[g_{\text{classical}}] + \frac{1}{\lambda} \; S_{\text{m}}[\chi, g_{\text{classical}}] + S_{\text{m}}[h] + \cdots$

AFT in curved spacetime

Small $C_\nu$ expansion breaks down at $\frac{E^2}{M^2} \approx O(1)$

For a sphere of radius $L$, $p$ is quantized as $\frac{1}{L}$

$\Rightarrow E^2 \sim p^2 \sim \frac{1}{L^2} \sim \nu$

$C_{\nu E^2}$
D. gravity in general dimensions

\[ S_{\text{grav}} = \frac{1}{16\pi G_d} \int d^d x \sqrt{-g} \ (R-2\Lambda) \]

\[ [G_d] = \frac{L^{d-1}}{M^2} \Rightarrow M_{\text{pl}}^d = \frac{\hbar}{G_d}, \quad l_{\text{pl}} = \hbar G_d \]

\[ \chi_G \sim \frac{C_d}{h^{d-3}} \sim \frac{E}{M_{\text{pl}}^2} \]

\[ r_s \sim (\gamma_{m,n})^{d-3} \]

consider \[ M_d = M_D \times Y \]

\[ \text{M}_d \text{ non-compact, } D\text{-dimensional} \]

\[ Y \text{ compact, } d\text{-D} \]

suppose \( Y \) is too small to be detected.

In \( \mathbb{R} \times S \): the effective Newton constant \( G_5 \)

for an observer is not the same as the fundamental

\[ \frac{1}{G_5} = \frac{1}{G_d} \frac{1}{V_Y} \frac{1}{\text{volume of } Y} \]
\[ l_{PD} = \frac{\frac{l_{pD}^{d-2}}{L^{d-D}}} \]

expect \( L > l_{pD} \)

\[ \Rightarrow l_{PD} < l_{pD} \]

Einstein gravity as E.F.T.

gravity tested to \( 10^{-2} \) cm

we're going down to \( 10^{-33} \) cm

1) Extra dimensions will see d-dim gravity before reaching \( l_p \)

2) String theory is string length

\[ \rightarrow \]

3) Suppose new physics appears at some scale \( L \sim \frac{1}{m} \)

\[ S = \frac{1}{2 \kappa_G} \int d^d x \sqrt{-G} \left[ R - 2 \Lambda + \frac{a_1}{m^2} R^2 + \frac{a_2}{m^2} R R W^2 + \cdots \right] \]
Consider a spherically symmetric, electrically neutral object of mass $M$.

The Schwarzschild solution (4D) can be analytically found to be:

$$ds^2 = -F dt^2 + \frac{1}{F} dr^2 + r^2 d\Omega^2$$

$$= g_{\mu\nu} dx^\mu dx^\nu$$

with

$$F = 1 - \frac{2GM}{r} = 1 - \frac{r_s}{r}$$

$$r_s = 2GM$$

Most important features:

1. $r \to \infty$, $F \to 1$, $g_{\mu\nu} \to g_{\mu\nu}$

2. $r = r_s$, $F \to 0$, $g_{tt} = 0$

$$g_{tt} = \infty$$

will see + (Schwarzschild time) becomes singular at $r = r_s$

$r = \text{const} > r_s \Rightarrow$ time-like hypersurface

$r = \text{const} < r_s \Rightarrow$ space-like hypersurface

$r = r_s \Rightarrow$ null hypersurface
3) $r = r_s$: event horizon

\[
\text{interior} \neq \text{exterior} \]

$r < r_s \quad r = r_s \quad r > r_s$

4) $r = r_s$: hypersurface of infinite redshift

consider an observer $O_h$ at $r = r_h \approx r_s$

\[
\text{proper time for } O_h \quad \text{at } r = \infty\]

\[
\text{proper time for } O_h \quad \text{at } r = \infty
\]

\[
d\tau_h = f^{1/2} \, dt = \left( -\frac{r_s}{r_h} \right)^{1/2} \, dt
\]

consider a physical process at $r = r_h$

with local proper energy $\varepsilon$

$O_\infty$ sees energy $E_\infty = \varepsilon \left( -\frac{r_s}{r_h} \right)^{1/2}$

as $r_h \approx r_s$, $E_\infty \to 0$ "infinite redshift"
\[ ds^2 = -f \, dt^2 + \frac{1}{f} \, dr^2 + r^2 \, d\Omega_2^2 \]

\[ f = 1 - \frac{r}{r_s}, \quad r_s = 2GM \]

causal structure & Rindler spacetime

consider \( r \geq r_s \), \( \frac{r-r_s}{r_s} \ll 1 \)

proper distance \( s \) from \( r = r_s \)

\[ d\rho^2 = \frac{dr^2}{f} \implies dp = \frac{dr}{f} \]

\[ \frac{dr}{f} \]

\[ f(r) = f(r_s) + f'(r_s) (r-r_s) + \ldots \]

\[ \rho = \frac{2}{\sqrt{f(r_s)}} \sqrt{r-r_s} \]

\[ f(r) = \left[ \frac{1}{2} f'(r_s) \right]^2 \rho^2 = \kappa \rho^2 \]

\[ ds^2 = -\kappa \rho^2 \, dt^2 + dp^2 + r_s^2 \, d\Omega_2^2 \]

\[ \eta = xt \]

\[ \text{Mink}_2 \quad (\text{Rindler spacetime}) \]

\[ ds^2 = -dt^2 + dx^2 = -\rho^2 \, dy^2 + dp^2 \]

\[ X = \rho \cos \eta \quad T = \rho \sinh \eta \]
\[ p^2 = x^2 - t^2 \]

\[ \tan \gamma = \frac{dx}{dt} \]

At \( x = T = 0 \) \( \Rightarrow p \rightarrow 0 \) \( \eta \) finite

At \( x = T \) \( \Rightarrow p \rightarrow 0 \) \( \eta \rightarrow -\infty \) \( \Rightarrow pe^{-\eta} \) finite

At \( x = -T \) \( \Rightarrow p \rightarrow 0 \) \( \eta \rightarrow +\infty \) \( \Rightarrow pe^{+\eta} \) finite

Rindler observers: \( p = \text{const} \) \( \Rightarrow r = \text{const} \)

\[ \frac{\partial \gamma}{\partial \eta} \Rightarrow \text{prop} = \frac{1}{p} \]

Note: No signal can propagate from \( F \) to \( R \)

\( X = T \): Future horizon (can only go in)

\( X = -T \): Past horizon (can only come out)

1) \( r = r_s \leftrightarrow p = 0 \leftrightarrow X = \pm T \)

null hypersurface

\((p,r)\) singular at \( p = 0 \) \& \((t,r)\) singular at \( r = r_s \)
2) \( r = \text{const observer} \)
\( \& \ p = \text{const rindler observer} \)

their accelerations agree

3) free-fall observer in BH \( \Leftrightarrow \) inertial observer in Rindler

4) Using \((T, X)\), we can extend the black hole geometry from \( r \gg r_s \) to \( \text{four regions} \) with the near-horizon metric

\[ ds^2 = -dT^2 + dX^2 + r_s^2 d\Omega^2 \]

\( T, X \) as coordinate transformation of \((r, t)\) and then extend them to \( \text{full spacetime} \)

(derive)

\[ ds^2 = g(r)(-dT^2 + dX^2) + r_s^2 d\Omega^2 \]

\[ g(r) = \frac{r_s}{r} e^{-\frac{r-r_s}{r_s}} \]

\( r \) should be considered as a function of \((r, T)\)

\[ X^2 - T^2 = \frac{r-s}{r_s} e^{-\frac{r-r_s}{r_s}} \]

a) \( g(r_s) = 1 \)

\[ X^2 - T^2 = 0 \]

\( r = r_s \)

b) singularity at \( r = 0 \)

\( \Leftrightarrow T^2 - X^2 = \frac{1}{p_{x^2}^2} > 0 \)
c) symmetries
   (i) $T \leftrightarrow -T$ $x \leftrightarrow -x$
   (ii) boost in $(T, X) \Leftrightarrow t \mapsto t + \text{const}

d) $L$ is a mirror of $R$ with another asymptotic flat region

e) $T=0$ slice

f) $F$: interior of BH
   (future horizon)

g) $P$: white hole
   (past horizon)

h) $L, P$ not present in collapse of a star

A digression: Penrose diagrams

Procedure:
1. choose a metric
   \[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]
   $x^\mu$ covers full spacetime

2. Find $x^\mu = x^\mu(y^a)$ s.t. $y^a$ has finite range

3. construct a new metric
   \[ ds^2 = \Omega^2(y)ds^2 = g_{ab}dy^ay^bdy^b \]
so that the causal structure of $\tilde{g}$ is known.

\[ ds^2 = -dT^2 + dX^2 \]
\[ = -dUdV \quad u = T - X \]
\[ v = T + X \]

\[ u = \tan u \Rightarrow u, v \in [-\frac{\pi}{4}, \frac{\pi}{4}] \]
\[ V = \tan V \]
\[ ds^2 = -\frac{1}{\cos^2 u \cos^2 v} \, du \, dv \]

\[ \Rightarrow \tilde{ds}^2 = -du \, dv \]

\[ \Rightarrow \]

\[ \eta_0 \]

\[ L \quad F \quad R \quad i_0 \]

\[ P \]

\[ L \quad F \quad R \quad P \]

Black Hole:

Kruskal coordinates $(X, T)$

stellar collapse:
Formulas we will use going forward

\[ ds^2 = -f \, dt^2 + \frac{1}{f} \, dr^2 + r^2 \, d\Omega^2 \]
\[ = -\kappa \rho^2 \, dt^2 + d\rho^2 + r_s^2 \, d\Omega^2 \quad \text{near horizon} \]

\[ \kappa = \frac{1}{2} \, \frac{f'(r_s)}{f(r_s)} = \frac{1}{2r_s} = \frac{1}{4GM} \]
1.3 Black Hole Temperature

1975: Hawking
1976: Unruh
Bisognano–Wichmann

Both phenomena at level of leading order in low-energy approx

QFT in a rigid curved spacetime
This effect is universal insofar as it would apply to any QFT regardless of interactions of matter content.

\[ S = -\int d^4x \sqrt{g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right] \]

e.g. (x)

1.3.1 Hawking & Unruh temperatures from Euclidean
analytic continuation

\[ S_\beta = \frac{1}{Z_\beta} e^{-\beta H} \]
with 

\[ Z_\beta = \text{Tr} e^{-\beta H} = \text{Tr}(e^{-iHt/\hbar}) \]

\[ ds^2 = -dt^2 + dx^2 \]
\[ + \rightarrow \text{e}^{-i2\pi t} - i2\pi - \rightarrow \text{e}^{i2\pi t} \]
\[ ds^2_E = dt^2 + dx^2 \]
\[ t = -i\beta \frac{t}{\hbar} \]
Thermal equilibrium at $T = \frac{1}{\beta}$ described by path integrals in (1) with periodicity $\beta T$

(A) Hawking temp: take BH metric

$$-\text{CT}$$

$$ds^2 = F dt^2 + \frac{1}{F} dr^2 + r^2 d\Omega^2$$
$$= \kappa^2 p^2 dt^2 + dp^2 + r_s^2 d\Omega^2$$
$$= p^2 d\theta^2 + dp^2 + r_s^2 d\Omega^2$$

Locally $\mathbb{R}^2$

Global structure: depends on periodicity $\theta$

$\theta \sim \theta + 2\pi$ $\Rightarrow$ globally $\mathbb{R}^2$

Other periodicity:

$\mathbb{S} \times \mathbb{S}$

$\Rightarrow$ ALE? gravitational instantons

$p = 0$ $\Rightarrow$ conical singularity

Smoothness & Euclidean geometry

$\Rightarrow \mathbb{C} \Rightarrow \mathbb{C} \times \mathbb{C}$ (uniquely determined)
Different from on Mink.

\[ R \times R^3 \rightarrow S' \times R^3 \]
\[ \uparrow \]
any period
allowed

\[ \Rightarrow \text{in a black hole geometry, quantum matter can be in equilibrium only at a single temperature} \]
\[ T_H = \frac{1}{\beta_H} \]

\[ \hbar \beta_H = \frac{2\pi}{\chi} \Rightarrow T = \frac{\hbar}{2\pi} = \frac{\hbar}{8\pi G M} \]

Remark:

1) \( T_H \) should be considered as temperature measured in proper units of proper time at \( r = \infty \).

2) \( \Rightarrow \) BH must have temperature \( T_H \).

3) Field theory on a cone
\[ \Rightarrow \] observables can be singular at the singularity.

4) suppose \( \tau \sim \tau + \beta H \) \( \beta < \beta_H \)

must be singular to screen difference

\[ T \neq T_H \]

\( p = 0 \) \( \Rightarrow \) horizon force the equilibrium.

\[ T_H \]

\[ T \neq T_H \]
5) You can put any matter at any $T$ outside the black hole (including nothing, $T=0$) → non-eq. state but euclidean A.C. you can only describe the equilibrium state.

6) $ds^2 = g(r) (-dt^2 + dx^2) + r^2 d\Omega^2$

$r = r (T^2 - X^2) \rightarrow T_E^2 + X^2 = T - i T_E$

7) For a stationary observer at $r$

$T_{loc} = \sqrt{r E(r)} dt$

$\Rightarrow T_{loc} = \frac{\hbar k}{2\pi} f^{1/2}(r)$ ②

$r \rightarrow r_3 \quad T_{loc} \rightarrow \infty$

B. Unruh temp

$ds^2 = -dt^2 + dx^2$

$\rightarrow ds^2 = -\beta^2 dy^2 + dp^2$

\[ ds^2 = \rho^2 d\theta^2 + dp^2 \]
Smoothness of Euclidean space
\[ \Theta = \Theta + 2\pi \]

local time:
\[ dt_{\text{loc}} = \gamma \, dt \]
\[ dx_{\text{loc}} = p \, d\Theta \]
\[ t_{\text{loc}} = t_{\text{loc}} + 2\pi \]
\[ x_{\text{loc}} = x_{\text{loc}} + 2\pi p \]
\[ T(u) = \frac{a \Theta}{2\pi} = \frac{T_0}{2\pi} , \quad a \mapsto \gamma \]

a uniformly accelerated obs. in Mink can be in thermal equilibrium only at
\[ T(u) \] otherwise one finds singular behavior at \[ T = \pm X (p=0) \]

Remarks:
1) 2) and 3) agree when \( \gamma = 1 \).

As expected

BH: \( r \to \infty \) \[ T \to T_H (a \text{prop} \to 0) \]

Rindler: \( p \to 0 \) \[ T \to 0 \] (a \text{prop} \to 0)
2) Does this happen to all accelerated observers?

\[ ds^2 = -g(r) \, dt^2 + \frac{1}{g(r)} \, dr^2 + r^2 \, d\Omega^2 \]

\[ g(r) = \frac{1}{-\frac{2GM(r)}{r}} \]

\[ m(r) = \begin{cases} \text{Mearth} & \text{if } r > r_e \\ \alpha r^3 & \text{if } r < r_e \end{cases} \]

\[ \Rightarrow g(r) \]

- Unruh, Unruh require \( g_{tt} \to 0 \)

3) Rindler \( T \neq T_u \)

Singular behavior at \( T = \pm X \)

1.3.2 Unruh temp. From entanglement:

1) Clarifies physical origin of temp.
2) Gives deeper understanding of the quantum state of matter
A digression — an alternative (Lorentzian) way to describe thermal states

\[ \rho = \frac{1}{Z_\beta} e^{-\beta H} \]

\[ \to \text{double the system} \]

\[ H_{\text{tot}} = H_1 \otimes H_2 \]

\[ H_1 \simeq H_2 \simeq H \]

**Typical state:**

\[ \sum_{m_n} a_{mn} \left( \ket{m_1} \otimes \ket{n_2} \right) \]

*non-factorizable*

\[ \equiv \Psi_1 \otimes \Psi_2 \]

\[ \to \text{entangled} \]

\[ \ket{\Psi} = \frac{1}{\sqrt{Z_\beta}} \sum_n e^{i \beta E_n} \left( \ket{n_1} \otimes \ket{\bar{n}_2} \right) \]

\[ \langle \Psi | \Psi \rangle = 1 \]

\[ \ket{\bar{n}} \text{ is } T \text{-reversal of } \ket{n} \]

\[ Z_\beta = \text{Tr}(e^{-\beta H}) \in \text{either system} \]
Consider $X$, which acts only on $\hat{q}$,
\[ \Rightarrow \langle \psi | X | \psi \rangle = \frac{1}{Z_B} \sum_{X/n} e^{-\beta E_X} \langle X | \psi \rangle \]
\[ = \text{Tr} (\rho_B X) \]
\[ \text{Tr}_2 (| \psi_B \rangle \langle \psi_B |) = i \rho_B \]

$| \psi \rangle$: thermal field double

Yonezawa (1968)

Remarks: Finite-$T$ effects come from:

1) Special entangled structure of $| \psi \rangle$
2) Ignorance of the other system
3) Purification of $\rho_B$
4) $(H_1 - H_2)(| \psi \rangle) = 0 \Rightarrow e^{-i(H_1 - H_2) t} | \psi \rangle = | \psi \rangle$
5) $H = \hbar \omega (a^\dagger a + \frac{1}{2})$ for harmonic oscillator

$| n \rangle = \frac{(a^\dagger)^n | 0 \rangle}{\sqrt{n!}}$

$\Rightarrow | \psi \rangle = \frac{e^{-\frac{1}{2} \beta \hbar \omega} e^{-\frac{1}{2} \beta \hbar \omega a^\dagger a}}{\sqrt{Z_B}} | 0 \rangle \otimes | 0 \rangle$
Recall: Path integral for vacuum state

\[ \Psi(x) = \langle x | 0 \rangle \]

\[ \Psi(x) \rightarrow -it \]

\[ = c \int \frac{Dx(t)}{x(t=0)} e^{-S_E [x(t)] / \hbar} \]

\[ = \lim_{0 \to -\infty} \langle x | e^{i\mathcal{H}t/\hbar} | 0 \rangle \]

\[ \psi_\lambda[\phi(x)] = \langle \phi(x) | 0 \rangle \]

\[ \phi(x=0, t) = \phi(x) \]

\[ = c \int \frac{D\phi}{\phi(t \to -\infty) \to 0} e^{-S_E[\phi] / \hbar} \]

\[ \langle x_2 | \mathcal{N}_2 = \frac{1}{2} (n_1 x_2) \rangle = \langle n | x_2 \rangle \]

\[ \Psi(x_1, x_2) = \langle x_1, x_2 | \Psi \rangle \]

\[ = \frac{1}{Z_0} \sum_n e^{-\frac{1}{2} \beta E_n} \langle x_1 | n \rangle \langle x_2 | n \rangle \]
\[ \sum_{\lambda} e^{-\frac{i}{2} \Theta_{\lambda}} \langle \lambda, \nu | \psi \rangle \langle \nu | \chi \rangle \\
= \langle \chi | e^{-\frac{i}{\hbar} \Delta \Phi} | \chi \rangle \\
= \langle \chi | e^{-\frac{i}{\hbar} \Delta \phi} | \chi \rangle \Delta \phi = \frac{i}{\hbar} \Theta \\
\sim = \frac{1}{Z_{\phi}} \int D\chi(2) e^{-\frac{i}{\hbar} S_{\phi}[\chi(2)]} \\
\sim = \frac{1}{Z_{\phi}} \int D\chi(2) e^{-\frac{i}{\hbar} S_{\phi}[\chi(2)]} \\
\Rightarrow \langle \chi' | \psi' \rangle = \frac{1}{Z_{\phi}} \int D\chi(2) e^{-\frac{i}{\hbar} S_{\phi}[\chi(2)]} \\
\Rightarrow \langle \chi' \mid \psi' \rangle = \frac{1}{Z_{\phi}} \int D\chi(2) e^{-\frac{i}{\hbar} S_{\phi}[\chi(2)]} \\
\text{assume } S_{\phi} \text{ invariant under } t \rightarrow -t \\
\Rightarrow \frac{1}{Z_{\phi}} \int D\chi(2) e^{-S_{\phi}[\chi]} = 1 \\
\Rightarrow \frac{1}{Z_{\phi}} \int D\chi(2) e^{-S_{\phi}[\chi]} = 1 \\
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\Rightarrow \frac{1}{Z_{\phi}} \int D\chi(2) e^{-S_{\phi}[\chi]} = 1 \\
\Rightarrow \frac{1}{Z_{\phi}} \int D\chi(2) e^{-S_{\phi}[\chi]} = 1 \\
\Rightarrow \frac{1}{Z_{\phi}} \int D\chi(2) e^{-S_{\phi}[\chi]} = 1 \\
\Rightarrow \frac{1}{Z_{\phi}} \int D\chi(2) e^{-S_{\phi}[\chi]} = 1
Field theory:
\[ \left( \Psi_1(x) \Psi_2(x) \right) \]
\[ = \frac{1}{Z_0} \int D\Phi(z,x) e^{-S_E[\Phi]} \]  
\[ \bar{\Psi}(x) = \Psi(x) \]

\[ \Psi(z,x) = \Psi(z) \]  \hspace{1cm} (*)

B. Unruh temperature from entanglement

\[ ds^2 = -d\tau^2 + dX^2 \]
\[ = -d\eta^2 + dp^2 \]
\[ X = p \cosh \eta \]
\[ \tau = p \sinh \eta \]

\[ \eta + \eta + \text{const} = \text{boost in } (T,X) \]

\[ \frac{\eta}{\gamma} \text{ similarly} \]

\[ X = -p \cosh \eta \]
\[ \tau = -p \sinh \eta \]

R, L are causally disconnected

Three sets of observers:

Mink: see the entire Mink space
Rind: \( R \) patch
Rind: \( L \) patch
Mink: Cauchy slice: $T = 0$
\begin{align*}
\Omega_{\text{Mink}} & : \text{span} \{ \phi(x) \}^\perp \\
\phi(x) & = \phi(T=0, x) \\
H_m & : \text{using } T \text{ as time}
\end{align*}

\begin{align*}
Rind: \text{ Cauchy slice: } & \eta = 0 \text{ (} X = 0 \text{ axis)} \\
\Omega_{\text{Rind}} & : \text{span} \{ \phi_R(p) \}^\perp \\
\phi_R(p) & = \phi(T=0, X=p=0) \\
H_R & : \text{obtained from } S \text{ restricted to } R \\
& \text{with } \eta \text{ as time}
\end{align*}

\begin{align*}
Rind_L: \text{ Cauchy slice: } & \eta = 0 \text{ (} X = 0 \text{ axis)} \\
\Omega_{\text{Rind}_L} & : \text{span} \{ \phi_L(p) \}^\perp \\
\phi_L(p) & = \phi(T=0, X=p<0)
\end{align*}
\[
\begin{align*}
\Phi(x) &= (\phi_L(x), \phi_R(x)) \\
\phi(L) &= (\phi_L(L)) \otimes (\phi_R(L)) \\
\triangleright \quad H_{\text{Mink}} &= H_{\text{Rind}} \otimes H_{\text{Rind}}
\end{align*}
\]

Question: Is \( H_{\text{Mink}} \) equivalent to \( H_{\text{Rind}} \otimes H_{\text{Rind}} \)?

Answer: It turns out, no.

Note: any field theory is CPT-invariant.

\[ H_R \leftrightarrow H_L \]

\((R, L) \) form a double

Claim: \( |0\rangle \) is a TFD for \( H_{\text{Rind}} \) and \( H_{\text{Rind}} \)

strategy for proof: coordinate space wavefunction
Note: \((T_E, X)\) - LHP in fact coincides with Euclidean analytic continuation of Rindler \(\eta + i\theta\)
\[\theta \in (\pi, 0)\]
\[\Rightarrow \psi_x[x] = \int \frac{dP_0(x^0)}{2\pi} \frac{e^{-s \mathcal{H}_E}}{Z_E} \]
\[\rho_0(x) = \rho_0(x)\]
\[\rho(\theta = -\pi) = \rho_L(\theta)\]
\[\rho(\theta = 0) = \rho_R(\theta)\]
\[\Rightarrow \psi_x[x] = \langle \rho_R(\theta) \rho_L(\theta) | \psi_x \rangle\]
\[\text{with } \frac{1}{2} \beta x = \pi\]
\[\Rightarrow \beta = \frac{2\pi}{\hbar}\]
\[\Rightarrow Z_{(Mink)} = Z_{(Rind)}\]
\[\Rightarrow Z_0 = Tr(e^{-2\pi \mathcal{H}_R})\]

We conclude:
\[M = \langle \psi_{\beta = \frac{2\pi}{\hbar}} \rangle\]
\[Z_0 = Tr(e^{-2\pi \mathcal{H}_{(Rind)}})\]
\[\text{since } \beta \text{ is associated with } y\]
\[dH_{\text{leak}} = \rho \, dy\]
\[\Rightarrow H_{\text{leak}} = \frac{2\pi \rho}{\hbar}\]
\[\text{just as we derived last time but this time for real-time wavefunction}\]
Remarks

1) Euclidean method: regularity of analytic continuation
   - only have equilibrium at $T_u$
   - When system is at $10^\circ$C
   - R/L observers both thermal at $T_u$
   \[ Z_0 = \frac{Z}{Z_0} = \frac{2}{Z_0} \]
   Mink Rind

2) Thermal nature comes from
   (a) Special entangled structure of $10^\circ$C
   (b) Tracing out the other half

3) Both derivations used a simple geometric feature:
   - Euclidean analytic cont. of Mink
   - Euclidean analytic cont. of Rind
   - Special periodicity

$\sim$ This is very general, applies to any RFT
4) Entanglement method: no need to deal w/ conical singularity

5) \((H_L - H_R) |\psi\rangle = 0 \Rightarrow e^{-i\mathcal{H} t} |\psi\rangle = (|\psi\rangle)

Boost inv. of vacuum \(\sim e^{-i\gamma (H_L - H_R) t} |0\rangle_m = (0)_{m}\)

\(H_R = \int_0^\infty dX \mathcal{H} \Theta_0 \) (on \(T=0\) Cauchy slice)

\(H_L = \int_{-\infty}^0 dX \mathcal{H} \Theta_0 \) (on "

C: Hawking temperature from entanglement

\[ ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2 \]

\[ X^2 - T^2 = \frac{1}{\lambda^2} e^{\frac{\lambda}{r_s} \left( \frac{r-r_s}{r_s} \right)} \]

Similarity: Kruskal observers \( \rightarrow T \)

Schwarzschild observers \( \rightarrow \frac{\lambda}{r} \)

\[ \mathcal{H}_K = \mathcal{H}_L \otimes \mathcal{H}_R \]

Important difference from Rindler

\( \rightarrow \) metric is not \( T \)-independent \( \Rightarrow \mathcal{H}_K \) is not either \( \rightarrow \) energy is not conserved

\( \sim \) no notion of vacuum state
Nevertheless we can define the counterpart of (10) using path integrals to get (10) "Hawking vacuum"

Key:
1. Kruskal metric allows a sensible $T \rightarrow iT$
2. Euclidean manifold is again the same as taking $t \rightarrow iT$ with $t \rightarrow t + \frac{i \pi}{a}$

$\Rightarrow \text{(def) } \langle \Psi_H \rangle_{HH} \text{ path integral over } T_E < 0$

$\langle \Psi_H \rangle_{HH} \rightarrow \text{Thermal Field double with } \beta_H = \frac{2\pi}{a}$

D. Geometry & Entanglement

Previously: From perspective of Rindler or Schwarzschild observers, there is a singular behaviour at the horizon unless they are at $T_H$.

$\rightarrow$ Explain this using entanglement

Rindler: $\eta \rightarrow \infty$ zero temp $\rightarrow \Theta$ not connect $\rightarrow 10^\lambda R \otimes 10^\lambda R$ (*)

In this state $L$ and $R$ are unentangled

For any smooth wavefunction of any QFT in Minkowski

$L \otimes R \rightarrow \text{always entangled for any finite energy state}$

For $L, R$ not entangled, we'd need a barrier $\partial KO$

$\Rightarrow$ singular behavior at $x = 0$ that causally propagates.
Note (*) is not a state in Mink₂
\[ L \otimes (R) \Rightarrow \text{No FP regions} \]
\[ \Rightarrow \text{Presence of F/P regions in Mink₂} \]
\[ \Rightarrow \text{Entanglement of L, R.} \]

Generalize:

1) Any finite energy state in Mink₂ requires specific entanglement between L and R

2) Generic state doesn't have that entanglement structure in \( \overline{\text{O ther}} \)
\[ \Rightarrow \text{Cannot be interpreted as sensible in Mink₂} \]

3) Similarly, generic state in \( \Omega \) \( \overline{\text{O ther}} \) for Schwarzschild will have "Fire wall" at the horizon

4) Discussion is at the level of states, independent of details about \( \Omega \), \( \overline{\text{M}} \), \( \overline{\text{H L, H R}} \), etc.
1.3.3 Free Field Theories

\[ |\psi \rangle = \frac{1}{\sqrt{2}} \sum_{n} e^{-\frac{i}{2} \sum_{n} n \theta} \Theta(n) = e^{-\frac{i}{2} \theta} \sum_{n} e^{\frac{i}{2} \theta} a_{n} a_{n}^{+} |10\rangle |10\rangle\]

Note: \[a |10\rangle = a e^{\alpha |10\rangle} \quad \alpha_{1} |10\rangle = 0 \quad \alpha_{2} |10\rangle = 0\]

\[\Rightarrow (a_{1} - e^{-\frac{i}{2} \theta} a_{2}^{+}) |\psi \rangle = 0\]
\[\Rightarrow (a_{2} - e^{-\frac{i}{2} \theta} a_{1}^{+}) |\psi \rangle = 0\]

Define:
\[b_{1} = \cosh \theta a_{1} - \sinh \theta a_{2}^{+}\]
\[b_{2} = \cosh \theta a_{2} - \sinh \theta a_{1}^{+}\]

\[\Rightarrow \text{take } \cosh \theta = \frac{1}{\sqrt{1 - e^{-2 \theta}}}, \quad \sinh \theta = \frac{e^{-\frac{i}{2} \theta}}{\sqrt{1 - e^{-2 \theta}}}\]

then \[\left[ b_{1}, b_{1}^{+} \right] = \left[ b_{2}, b_{2}^{+} \right] = 1\]
else \[\left[ , \right] = 0\]

and \[b_{1} (|\psi \rangle) = b_{2} (|\psi \rangle) = 0\]

\[\Rightarrow |\psi \rangle \text{ is vacuum for } b_{1}/b_{2}\]

Free Field Theories \(\rightarrow\) a bunch of harmonic oscillators

\[|0\rangle_{1}, |0\rangle_{2} \rightarrow |0\rangle_{1} |0\rangle_{R}, |\psi \rangle \rightarrow |10\rangle_{m}\]
Free massless scalar:

\[ S = -\frac{1}{2} \int d^4x \, \partial_\mu \phi \partial^\mu \phi \]

Minkowski observers

\[ (-\Delta + m^2) \phi = 0 \]

\[ \psi_{\text{Mink}} = \frac{1}{\sqrt{2w_{\psi}}} e^{-i\omega_{\psi} \tau + ipx} \]

\[ w_{\psi} = \frac{1}{w_{\psi}} \]

\[ U = T - \chi \Rightarrow \psi_{\text{Mink}} = \frac{1}{\sqrt{2w_{\psi}}} e^{-i\omega_{\psi} \tau - \frac{1}{2} \Delta x^2 - ipx} \]

\[ V = T + \chi \]

\[ \{p_x, p_y\} = i \delta(p_x - p'_x, p_y - p'_y) \]

\[ \{u_x, u_y\} = 2\pi \delta(p_x - p'_x) \]

\[ \{u_x^*, u_y^*\} = 2\pi i \delta(p_x - p'_x) \]

\[ \{u_x^*, u_y^*\} = 0 \]

CCR with \( \theta = \sum_p \left( \psi_p u_p + \psi_p^* u_p^* \right) \)

\[ [\psi_p, \psi_p^+] = 2\pi \delta(p - p') \]

\[ a_p \hat{a}_p^+ |0\rangle_p = 0 \]

Rindler "R"-observers

\[ ds^2 = -dT^2 + dx^2 = -p^2 dy^2 + dp^2 \]

\[ p = e^{-\frac{\chi}{2w_{\psi}}} \Rightarrow ds^2 = e^{2\frac{\chi}{w_{\psi}}} \left( -dy^2 + dx^2 \right) \]

\[ (\Delta^2 + m^2) \phi = 0 \]

\[ U = \eta - \chi \Rightarrow \phi = \frac{1}{\sqrt{2w_{\phi}}} e^{-i\omega_{\phi} \tau - \frac{1}{2} \Delta x^2 - ipx} \]

\[ V = \eta + \chi \]

\[ u_x = e^{-\frac{\chi}{w_{\phi}}} \]

\[ v_x = e^{-i\omega_{\phi} \tau} \]

\[ \phi_R = \sum_k (b_k^R \psi_k + b_k^{R*} \psi_k^*) \]

\[ [b_k, b_k^+^*] = 2\pi \delta(k - k') \]

\[ b_k^R |0\rangle_R = 0 \]
Sept 26 (Missed, got notes from Sam)

Recall \( U = e^{-u}, \ V = e^u \) (So \( U < 0, \ V > 0 \)

\[ V_k = \begin{cases} \frac{1}{\sqrt{2w_k}} e^{-i\omega_k u} & k > 0 \\ \frac{1}{\sqrt{2w_k}} e^{i\omega_k u} & k < 0 \end{cases} \]

\[ \mathcal{P} = \sum_k b_k^{(R)} V_k + b_k^{(R^*)} V_k^* \Rightarrow \begin{bmatrix} b_k^{(R)} & b_k^{(R^*)} \end{bmatrix} = 2\pi S(k\_R) \]

Define right vacuum \( b_k^{(R)} \big| 0^R \big> = 0 \)

Note \( \langle \mathcal{P}_R \rangle, \langle \mathcal{P}_R^* \rangle = -i \int_0^\infty \langle R \big| \mathcal{P} \big| 0^R \rangle \langle \mathcal{P}^* \big| 0^R \big> \frac{d\omega}{\omega} \)

\( V_k \) is singular at \( U = 0 \) and \( V = 0 \) since the modes are only supported in the \( R \) region. Potential singular behavior in physical quantities.

Rindler \( \mathcal{L} \):

\[ T = -e^{-u} \cosh y, \ X = -e^{-u} \cosh y, \ U = e^u, \ V = -e^u \]

\[ V_k = \begin{cases} \frac{1}{\sqrt{2w_k}} e^{-i\omega_k u} & k > 0 \\ \frac{1}{\sqrt{2w_k}} e^{i\omega_k u} & k < 0 \end{cases} \]

PT reversal ensures \( k > 0 \) has \( U \), \( k < 0 \) has \( V \)

Note: \( \langle \mathcal{P}^{(2)} \rangle, \langle \mathcal{P}^{(1)} \rangle = -i \int_0^\infty \langle \mathcal{P} \big| \mathcal{P} \big| 0^R \rangle \langle \mathcal{P}^* \big| 0^R \big> \frac{d\omega}{\omega} \) (recall for \( k \_R \))
So \( P_L = \sum_k (b_k^{(L)} w_k + b_k^{(R)} w_k^*) \Rightarrow [b_k^{(L)}, b_k^{(R)*}] = 2\pi \delta(k) \).

Left vacuum \( b_k^{(L)}/\hbar \chi = 0 \).

On Minkowski, \( \Phi(T, X) = (\Phi_L, \Phi_R) \).
\n\[ \Rightarrow \Phi = \sum \{ a_p u_p^+ + a_p^+ u_p \} = \sum \{ b_k^{(L)} w_k + b_k^{(L)*} w_k^* + b_k^{(R)} v_k + b_k^{(R)*} v_k^* \} \]

To find relation between \( \delta a_p, u_p^+ \) and \( \delta b_k^{(L)}, b_k^{(L)*}, b_k^{(R)}, b_k^{(R)*} \), need to find relations between \( \delta u_p, u_p^* \) and \( \delta w_k, w_k^*, v_k, v_k^* \).

Two possibilities:

1. \( v_k = \sum P_c p u_p, w_k = \sum c_p u_p \) without \( u_p^* \) involved.

   \( \Rightarrow \) positive frequency modes of both \( L \) and \( R \) observers are related to only positive freq. Minkowski mode.

   \( \Rightarrow b_k^{(R)} = \sum \Delta p k a_p, b_k^{(L)*} = \sum \Delta p k a_p^* \)

   \( \Rightarrow \Phi \mid_m \) coincides with \( \Phi_L \otimes \Phi_R \).

2. suppose \( u_p = \sum_k \{ \Delta p k v_k + \Delta p k w_k + e_k v^* + e_k w^* \} \)

   then \( b_k^{(R)} = \sum \{ \Delta p k a_p^* + e_k^* a_p \} \), \( b_k^{(L)} = \sum \{ \Delta p k a_p + e_k a_p^* \} \)

\( \Rightarrow \Phi \mid_m \neq \Phi_L \otimes \Phi_R \).
Requiring \( b^{(R)}_k, 10^L \otimes 10^R = b^{(L)}_k, 10^L \otimes 10^R = 0 \)
we get:
\[
10^L \otimes 10^R \sim e^{water} 10^M
\]
\[
10^M \sim e^{symm^+} 10^L \otimes 10^R
\]
Focus on right-moving modes:
\[
\nu_p = \frac{1}{\sqrt{wp}} e^{-iwp} \quad \nu_p = \begin{cases} 1 & \text{for } u > 0 \\ \frac{1}{\sqrt{wp}} e^{iwp (\log - w)} & \text{for } u < 0 \\ 0 & \text{for } u = 0 \end{cases}
\]

it is analytic in lower complex \( \nu \) plane
\rightarrow SO is any linear superposition
Neither \( \nu_k \) or \( \nu_k^+ \) is analytic there
\rightarrow \nu_k, \nu_k^+ must involve both \( \nu_p, \nu_p^+ \)

Instead of finding \( \omega_k, \omega_k^+, \nu_k, \nu_k^+ \) explicitly,
consider another basis equivalent to \( (\nu_p, \nu_p^+) \)
i.e., has the same vacuum but related to \( \nu^+, \nu^+, \nu^+ \) in a simple way:

instead of finding \( \omega_k, \omega_k^+, \nu_k, \nu_k^+ \) explicitly,
consider another basis equivalent to \( (\nu_p, \nu_p^+) \)
i.e., has the same vacuum but related to \( \nu^+, \nu^+, \nu^+ \) in a simple way:
\[ \text{\( \xi \lambda_k, X_k \) share the same vacuum, \( \text{10} \)_M as \( \text{2\mu} \) \}

\[ V = \sum_k \left[ c_k \lambda_k + d_k X_k + \text{h.c.} \right], \quad c_k \langle 0 \text{10} | = d_k \langle 0 \text{10} | = 0 \]

\[ C_k = \cosh \theta_k \ b_k^{(R)} - \sinh \theta_k \ b_k^{(L)} \]

\[ D_k = \cosh \theta_k \ b_k^{(L)} - \sinh \theta_k \ b_k^{(R)} \]

where \( \text{cosh} \theta_k = \frac{e^{\frac{k}{k}}}{\sqrt{2 \sinh \theta_k k}} \)

\[ \text{sinh} \theta_k = \frac{e^{\frac{k}{k}}}{\sqrt{2 \sinh \theta_k k}} \]

\[ \Rightarrow \text{10} \langle 0 \text{10} | = \prod_k \left( \frac{e^{\frac{k}{k}}}{\sqrt{2 \sinh \theta_k k}} \right)^2 \exp \left[ \sum_k e^{-\pi k \theta_k} b_k^{(R)} b_k^{(L)} \right] \]

with \( Z_k = \frac{1}{2 \sinh \theta_k k} \)

\[ \text{(Free) Massive scalar in Schwarzschild background} \]

\[ \text{(HH) \quad \text{Squeezed state for Schwarzschild obs.}} \]

\[ \text{Realistic BH} \]

\[ \text{10}_M \]

\[ \text{get temp. from free field theory} \]
BH Thermodynamics

BH has a temperature

\[ T_H = \frac{kT}{2\pi} = \frac{T}{8\pi G N} \]  (1)

natural to interpret it as a thermodynamic system

suppose it has an entropy \( S \).

Expect it to satisfy 1st law:

\[ dE = T dS \]  (2)

identify \( E = MA \), integrate (2) to find \( S \)

\[ dS = \frac{dM}{T} \Rightarrow S = \frac{4\pi r_s M^2}{k} \]

but \( r_s = 2G_M \)

\[ \Rightarrow S = \frac{4\pi r_s^2 M}{GN} = \frac{A}{4\pi G_N} \]  (3)

So using (1) and (3), we can rewrite (2) as

\[ dM = \frac{kT A}{2\pi} = \frac{T}{8\pi G_N} dA \]  (4)

(4) is a pure geometric relation

Eq. (4) is part of a set of four laws on general BHs called "Four laws of BH mechanics"
No hair theorem:
A stationary asymptotically flat BH is solely characterized by:

1) mass $M$
2) angular momentum $J$
3) electric or magnetic charges $Q$

Four laws: (1972)

0th law: Surface gravity $\kappa$ is constant over the horizon

1st law: $dM = \frac{\kappa}{8\pi G_0} dA + Q dJ + \Phi dQ$

$Q$: angular frequency at the horizon
$\Phi$: electric potential (s.t. $\Phi(\infty) = 0$)

2nd law: Horizon area never decreases classically.

3rd law: Surface gravity of a BH cannot be reduced to 0 in a finite sequence of operations

With (1) + (3), the Four laws of BH mechanics become the standard laws of thermodynamics
Beckenstein (1972-1974):

BH should have an entropy \( A \) otherwise the second law of thermodynamics would be violated in the presence of a BH.

\[ S_{\text{tot}} = S_{\text{matter}} + S_{\text{BH}} \]

\[ \rightarrow \text{Generalized second law} \]

\[ \Delta S_{\text{tot}} \leq 0 \]

with (1975) Hawking radiation. GSL becomes standard 2nd law.

Remarks:

1) Classical limit \( t \to 0 \) (\( \text{Gen} \) fixed)

\[ T_H \to 0, \quad S_{\text{BH}} \to \infty \]

2) \[ T_H \propto \frac{1}{M}, \quad M \to T_H \]

\[ C = \frac{2M}{2T} = -\frac{T}{8\pi G N} \frac{1}{T^2} < 0 \Rightarrow \text{negative specific heat} \]

\[ \Rightarrow T_H \text{ not a stable equilibrium. Why? Consider small fluctuations in } T_{\text{BH}}, \text{ say } T_{\text{BH}}^\uparrow \text{ radiate a bit more to env + ML } \to T_{\text{BH}}^\downarrow \text{ radiate even more and similarly unstable in the other direction.} \]
BH + infinite bath

stable equilibrium is possible in a finite box

smaller and smaller

BH bigger and bigger

3) \[ T_H = \frac{\hbar x}{2\pi} \quad \text{and} \quad S = \frac{A}{4\pi G_N} \]

universal

Apply to any matter coupled to Einstein gravity. (AdS, dS, Mink, all spacetimes)

4) With higher derivative corrections to Einstein gravity
(i.e. \( R^2 + x^2 R_{\mu\nu}^2 + x^2 R^2 \ldots \))

these rules no longer apply, but \( S, T_H \) can

still be expressed in terms of horizon quantities
1.5 Quantum Nature of Black Holes and the Holographic Principle

BH thermodynamics + TH at

\[ S_{BH} \propto k \]

\[ T \]

→ Natural to treat BH as a macroscopic quantum statistical system.

Questions:

(1) What is the statistical interpretation of the entropy of a black hole?

From standard stat mech:

\[ \Omega \text{ microstates} \]

\[ \Rightarrow S = k_B \log \Omega \]

For BH, expect \( \Omega = e^{A/4\pi G_0} = e^{A/4\pi} \)

Heuristically:

\[ \text{entropy} \sim \text{put 1 d.o.f. in each Planckian cell.} \]

(e.g. spin)
(a) This is a huge entropy
For $M_{BH} = M_{\odot}$, $r_s = 3\text{km}$, $\frac{A}{\hbar^2 G m} \approx 1.1 \times 10^{57}$
$\Rightarrow \Omega \approx e^{10^{57}}$

The sun itself has entropy $\frac{s}{k} \approx 10^{57}$

(b) When a star collapses to form a BH, there is a huge increase in available microstates.

No hair theorem: all these states must be quantum mechanical in nature.

Huge increase $\Leftrightarrow$ gravity is (Planck scale is very small)

(c) In string theory, there are black holes whose microstates can be precisely counted, giving (after complicated combinatorics), exactly an entropy $S = \frac{A}{\sqrt{\hbar G m}}$

(d) In holographic duality, for black holes in this, the statistical origin is again known
Hawking's information loss paradox

1. To an excellent approximation BH radiates thermally for $M > M_{Pl}$ 
   "white noise"

2. BH loses mass

3. Should disappear

But when $M \sim M_{Pl}$, not enough d.o.f. to encode all the information put into it.

Another way to say this:

Suppose a star is in a pure state

$\Rightarrow$ BH

$\Rightarrow$ Radiates

$\Rightarrow$ Radiation (mixed thermal density matrix)

$\Rightarrow$ 3 logical possibilities:

1. Information is lost $\Rightarrow$ QM must be modified

2. Hawking radiation stops at $M = 0(M_{Pl})$
   $\Rightarrow$ Planckian mass remnant is left, which encodes all information

3. No remnants, unitary evolution $\Rightarrow$ information comes out from radiation
1) Is the most radical. It is also fiendishly difficult to modify QM.

2) Blames unknowns on another universe.
   A variant:
   \[ \triangle \]

3) Is the most conservative.
   \[ \Rightarrow \] significant challenges still to explain how the information comes out from radiation.
   \[ \Rightarrow \] imply quantum gravity puts highly nontrivial constraints/implications on low-energy physics.
   Simpler question: Burning of coal preparation: A typical highly excited pure state in a non-integrable many-body system looks thermal if one only probes a small part of it.

Say I separate the system in two:

\[ A + B \]

\[ P_A = \text{Tr}_B (|\Psi\rangle \langle \Psi|) \]

is very close to being thermal provided \( |A| \approx |B| \).
One reveals a given state is pure only by having full global information.

some remarks:

(a) At a given time, emitted photons look almost perfectly thermal.

(b) Nevertheless, they do contain information, but in a very subtle way. The information is encoded in the entanglement with the rest of the system.

(c) Consider the following quantities:

- Thermal entropy of photon gas: $S_{\text{gas}}$
- Thermal entropy of the coal: $S_{\text{coal}}$
- Entanglement entropy of photon gas: $S_{E_{\text{gas}}}$
- Entanglement entropy of the coal: $S_{E_{\text{coal}}}$

$c_{T}$ “coarse-grained” $S_{\text{gas}}$
$c_{E}$ “fine-grained” $S_{\text{gas}}$
$c_{E_{\text{coal}}}$

$S_{(PA)} = -Tr(g_{A} \log g_{A})$
This is a good paradigm for BH evaporation but BH is not coal.
Coal is causally connected with emitted radiation
Black hole's falling matter is causally disconnected with the emitted radiation.

Would either violate
  - No-cloning theorem (QM)
  - Locality (QFT)

Firewall?

Left Universe

Right Universe

perfectly
entangled

Holographic duality tells us that information evaporation for a Black Hole should be just like the burning of coal
Entropy bounds and holographic principle
starting point:
BH is a quantum statistical system
+a couple of “facts”
⇒ entropy bounds and holography

Facts:
(1) A sufficiently massive object in a compact volume always collapses to form a Black Hole
Rule of thumb: \(2G\pi M > L \Rightarrow BH\)
(2) Entropy reflects \(\approx\) degrees of freedom
\(\rho \rightarrow S = -\text{Tr} [\ln \rho]\)
⇒ for a system with \(N\)-dimensional Hilbert space
\(S_{\text{max}} = \log N\)
(a) For a system of spins \(N = 2^n \Rightarrow S_{\text{max}} \sim n\)
(b) \(H\) for h.o. is infinite-dimensional, but for finite energy, \(H\) is f.d.

* Spherical entropy bound
⇒ Take an isolated system of energy \(E\), entropy \(S_0\) in asymptotically flat spacetime

Let \(A\) be the area of smallest sphere enclosing the system, \(M_\text{BH}\) be the mass of a black hole with that area.
Then $E = Ma$

Maximal energy one could add (keeping $\gamma$ fixed) is $M_a - E$.

$S_{\text{final}} = S_{\text{BH}} \geq S_{\text{init}} \geq S_0(S')$

$S_0 \leq S_{\text{BH}} = \frac{A}{4\pi \hbar c a} \Rightarrow S_{\text{max}} = \frac{A}{4\pi \hbar c a}$

Remarks:

1) $S$ is classically extensive $\propto V$

$\neq$ QC behaves very differently from non-gravitational systems.

2) A cubic lattice of spins (size $L$, spacing $a$) has $S_{\text{max}} = \frac{L^3}{a^3} \log 2$ $(\ast)$

At $G = 0$ $S_{\text{max}} = 0$

Now slowly increase $G = G(aL)$

Then $(\ast)$ violates the bound when

$$\frac{L^3}{a^3} \log 2 > \frac{A - L^2}{4\pi \hbar c a L^2} $$

$$\frac{L^2}{a^2} > \frac{a}{L} \Rightarrow \lambda = L$$
Suppose each site has mass $m$

$$M = \frac{L^3}{\Lambda^3} m$$

Total system is a black hole when

$$\frac{\hbar c}{L} M > L \Rightarrow \frac{\hbar^2}{\Lambda^3} \frac{L^3}{m} \frac{m}{\hbar c} > L$$

$$\Rightarrow \frac{\hbar^2}{a^2} > a \frac{L}{L}$$

$$x_2 \leq x_1$$

So $\omega \propto A$ and $\omega \propto V$

But both conditions are important.

a) In a closed universe $S^3$

$$A = 0, \quad S \neq 0$$

trivially violated

b) Consider non-spherical region in asymptotically flat universe

Take $V$ space-like, with boundary $\partial V$

$$S_{max} = \frac{A(\partial V)}{4\pi}$$

(a) + (b): For a general region, $V$, with $\partial V$ in general $\partial V$ has not much to do with physics inside $V$

I don't understand this.
A generalization for asymptotic flat spacetime (universe)

Consider a general codimension 1 spacelike closed surface

\[ B \]

Future in, out

Past in, out

\[ \text{cf. hep-th/0203101} \]

\[ \Rightarrow \] light rays

\[ \Rightarrow \] causal diamond of \( B \)

Take \( D \) to denote the causal diamond

any point in \( D \) is fully determined by the information enclosed in \( D \)

conjecture:

Entropy on any cauchy slice of \( D \) \( \leq \frac{A_B}{4\pi G N} \)

This doesn't work in cosmological settings or inside a black hole.

Most general formulation:

For any \( B \), construct light-sheet

From \( B \): null hypersurface formed by non-expanding light rays from \( B \).

\[ \Rightarrow S[L(B)] \leq \frac{A_B}{4\pi G N} \]

\[ \text{entropy of "matter" d.o.f. passing through light sheet.} \]
Entropy bound - entropy associated with *d.o.f.*

⇒ statement on *d.o.f.*

⇒ Holographic principle

"When something is so new, that you don't have the language to describe it, you describe it in whatever language you can. It may not be precise, it may not even be true, but it's better than nothing."

~ Xiao-gang Wen

⇒ A spherical region of boundary area $A$ can be fully described by no more than

$$A = \frac{A}{4\pi a^2} \text{ d.o.f.}$$

i.e. one degree of freedom per planck-area