$6j$ symbols and the Tetrahedron

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1 3-j Symbols

Remember that for two irreducible representations of $SU(2)$ we had that $V_a \otimes V_b$ decomposes into a direct sum

$$V_{a+b} \oplus V_{a+b-2} \cdots \oplus V_{|a-b|}.$$

We want to know if a given representation V_c appears in $V_a \otimes V_b$, but this the same as asking if $\hom(V_c, V_a \otimes V_b)$ contains an $SU(2)$ invariant element. But this is canonically isomorphic to $V_a \otimes V_b \otimes V_c^*$ which is again, canonically isomorphic to $V_a \otimes V_b \otimes V_c$. So V_c appear in $V_a \otimes V_b$ iff the $SU(2)$ invariant V_0 appears (exactly once) in this triple product.

In particular $V_a \otimes V_b$ has an $SU(2)$ invariant part iff $a = b$. (interpretation physically). More generally, we need a, b, c to satisfy a triangle inequality. If we have $v_j \in V_c(j)$ then we can rewrite this as $v_j = \sum_{j',j''} c_{j,j',j''} v_{j'} \otimes v_{j''}$. These cs are unique (up to an overall scalar) and are called the $3j$ symbols.

2 Introduction to 6-j Symbols

Now note that because tensoring is canonically associative:

$$V_k \otimes (V_n \otimes V_m) \cong (V_k \otimes V_n) \otimes V_m$$

We can write the left hand side as

$$\cong \bigoplus_{\ell=|n-m|}^{n+m} V_k \otimes V_\ell \cong \bigoplus_{s=|k-\ell|}^{k+\ell} \bigoplus_{\ell=|n-m|}^{n+m} V_s$$

where the $\bigoplus_i$ is with a step size of 2, not 1. On the other hand, the right hand side is similarly:

$$\cong \bigoplus_{r=|k-n|}^{k+n} V_r \otimes V_m \cong \bigoplus_{s=|r-m|}^{r+m} \bigoplus_{r=|k-n|}^{k+n} V_s.$$

So we can express vectors in this triple tensor space either in terms of the weight basis associated with a set of weight spaces $|k, n, m, \ell, s\rangle$ or the weight basis $|k, n, m, r, s\rangle$. In quantum physics, these would be the eigenbasis for measurements of the three particles, coupling 1-2, and total spin and the eigenbasis for measurements of the three particles, coupling 2-3, and total spin, respectively. The change of basis is given by:

$$|j_1, j_2, j_3, j_{12}, J\rangle = \sum_r \langle j_1, j_2, j_3, j_{23}, J | j_1, j_2, j_3, j_{12}, J\rangle \, |j_1, j_2, j_3, j_{23}, J\rangle$$

The $6j$ symbols are then defined as:

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23} \end{Bmatrix} = (-1)^{(j_1+j_2+j_3+J)/2} \frac{\langle j_1, j_2, j_3, j_{23}, J | j_1, j_2, j_3, j_{12}, J\rangle}{\sqrt{(j_{12}+1)(j_{23}+1)}}$$

note, above, we did not use physics convention for the j_i, but instead mathematical representation theoretic convention (with the factor of 2). The strange square roots in the denominator serve to put j_{12}, j_{23} on equal footing with the rest of the j_i, so that the $6j$ symbols can obtain their notable tetrahedral symmetry structure.

3 Action on Homogenous Polynomial Spaces

From before, we know V_a for $SU(2)$ acts on the space of degree a polynomials on $\mathbb{C}^2$. We know there is an $SU(2)$ invariant part of $V_a \otimes V_a$, and we can call this ε^{aa}. If we view the representation as acting on polynomials, we get that this element is a polynomial on $\mathbb{C}^2 \oplus \mathbb{C}^2$:

$$(Z_1 W_2 - W_1 Z_2)^a$$

where we have raised to the power a to ensure its a degree $2a$ polynomial on which $V_a^{\otimes 2}$ can act. Now from before we know that the $SU(2)$ invariant part of $V_a \otimes V_b \otimes V_c$ is either dimension one or trivial from the triangle conditions. The invariant element ε^{abc} is a little bit trickier here. $SU(2)$ acts on $\mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2$. The elements like $Z_1 W_2 - W_1 Z_2$, $Z_1 W_3 - W_1 Z_3$, and $Z_2 W_3 - W_2 Z_3$. We need the powerso of Z_1, W_1 to have degree a, Z_2, W_2 to have degree b and Z_3, W_3 to have degree c. The right way to do this is to write:

$$\varepsilon^{abc} = (Z_1W_2 - W_1Z_2)^{(a+b-c)/2}(Z_1W_3 - W_1Z_3)^{(a+c-b)/2}(Z_2W_3 - W_2Z_3)^{(b+c-a)/2}$$

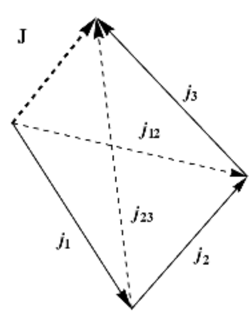
We rescale ε^{aa}, etc. to have norm $\sqrt{a+1}$ and ε^{abc} to have norm 1. Now if we have six irreps, $V_a, \ldots, V_f$, then $u = \varepsilon^{aa} \otimes \cdots \otimes \varepsilon^{ff}$ exist within a 12-fold tensor product and $v = \varepsilon^{abc} \otimes \varepsilon^{cde} \otimes \varepsilon^{efa} \otimes \varepsilon^{fdb}$ also exists there.

We can then take the inner product of these two $SU(2)$-invariant vectors and obtain:

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = (-)^{a+b+c+d+e+f}(u, v)$$

Note these are invariant under the action of the tetrahedral group A_4.

We are going to consider how three particles can combine angular momenta to form a final system. In the following diagram j_1, j_2, j_3 combine to form J. In particular, given that j_1 and j_2 combine to get j_{12} then we can also ask how can j_1, j_3 combine to get j_{13}... this gives the following tetrahedron.



The $6j$ symbol associated with it can be viewed as a sort of matrix element, and just as quantum mechanics always predicts, the square of a matrix element is a associated to a probability. In this case, it is the probability that given j_1, j_2 combine to get j_{12}, we have j_1, j_3 combine for j_{13}

4 The Formula

This paper explores an asymptotic formula for the $6j$ symbols, which was before noticed by physicists Ponzano and Regge (without proof, because they're physicsts).

$$\begin{Bmatrix} ka & kb & kc \\ kd & ke & kf \end{Bmatrix} \approx \sqrt{\frac{2}{3\pi V k^3}} \cos\left(\sum_a (ka+1)\frac{\theta_a}{2} + \frac{\pi}{4}\right)$$

for euclidean tetrahedra, and "exponentially decaying" for icosahedral ones.

Wigner used some reasoning to expect the square of the $6j$ symbols, the physical observable probability discussed above, to go as

$$\begin{Bmatrix} ka & kb & kc \\ kd & ke & kf \end{Bmatrix}^2 \approx \frac{1}{3\pi V}$$

There are very fast oscillations in the $6j$ symbols, so this represents a sort of "local average" rather than an honest-to-god approximation.

5 Some Background

What makes a Manifold M Kahler? Let it be $2n$ real dimensional, and on each tangent space $T_p(M)$ it posseses a real-valued Riemannian metric B and symplectic form ω. We let them be J invariant by: $B(JX, JY) = B(X, Y)$ and $\omega(JX, JY) = \omega(X, Y)$. We also relate them by: $B(X, Y) = \omega(X, JY)$

Then since ω is antisymmetric while B is symmetric we get $h = B(X, Y) - i\omega(X, Y)$ is now a bilinear form on $T_p(M)$ mapping into $\mathbb{C}$ that is *Hermitian* in X, Y.

We call a bilinear form $h : V \times V \to \mathbb{C}$ Hermitian if $h(w, z) = \overline{h(z, w)}$

If G acts symplectically on M (it preserves ω), that is the vector fields X_g have $\mathcal{L}_{X_g}\omega = 0$. Then we have ways of going between elements $\xi \in \mathfrak{g}$ defining vector field X_ξ and a Hamiltonian $\mathcal{H} = \mu(\xi)$ so that $dH = \iota_{X_\xi}\omega = \omega(X_\xi, -)$. Obtaining a Hamiltonian from a given vector field is a well-known process, and we can write $\mathcal{H} = \mu(X_\xi)$ sometimes.

The $k+1$ irrep of $SU(2)$ V_k is obtained from $S^2 = \mathbb{P}^1$ by giving it a round metric and a hermitian line bundle $\mathcal{L}^{\otimes k}$ (meaning the space of all homogenous polynomials of Z, W of degree k). On the Lie algebra $U(1)$ of the circle choose ξ so that we have $X_\xi = 2\pi\frac{\partial}{\partial\theta}$, i.e. $e^\xi = 1$.

6 Stationary Phase Formula

For a complex function ϕ with isolated critical points on a manifold of complex dimension n and volume form Ω, we have that the value of the integral:

$$\int_M e^{k\psi}\Omega \approx \left(\frac{2\pi}{k}\right)^n \sum_{p\ critical} \frac{e^{k\psi(p)}}{\sqrt{-\mathrm{Hess}_p(\psi)}}$$

7 Example Calculation

7.1 Finding the norm of s^k

Take the section $s^k : (Z, W) \mapsto Z^k W^k \in \mathbb{C}$, and in the complement of infinity: $(0, 1)$, we can trivialize $W \neq 0$. So let our coordinate by $\zeta = Z/W$. The pointwise norm at ζ of b^{2k} is given by taking $(\zeta, 1)$ and normalizing to give $\frac{(\zeta,1)}{\sqrt{1+|\zeta|^2}}$, which under b^{2k} gets mapped to $\frac{1}{(1+|\zeta|^2)^k}$, and similarly the pointwise norm of s^k here is $\frac{|\zeta|^k}{(1+|\zeta|^2)^k}$.

Using stereographic projection we can write $\zeta = \frac{x+iy}{1-z}$, $1/(1 + |\zeta|^2) = (1 - z)/2$ and $|\zeta|^2/(1 + |\zeta|^2) = (1 + z)/2$. We then have that at a point, $\langle s^k, s^k\rangle = \frac{|\zeta|^{2k}}{(1+|\zeta|^2)^{2k}} = \left(\frac{1-z^2}{4}\right)^k$. Now what's the global square of the norm?

$$||s^k||^2 = \int_{S^2} \left(\frac{1-z^2}{4}\right)^k \Omega = \int_{S^2} \left(\frac{1-z^2}{4}\right)^k 2k\frac{1}{4\pi} d\theta \wedge dz$$
$$= 2k \int_{-1}^{1} \frac{1}{2} \left(\frac{1-z^2}{4}\right)^k dz = 2k\frac{\Gamma^2(k+1)}{\Gamma(2k+2)}$$

Using Sterling's approximation, this goes as $\sqrt{\pi k}\ 4^{-k}$

7.2 Matrix elements of $SO(3)$ under rotation

Let V_{2k} be an irrep of $SO(3)$ (only even ones can be), and let S^1_z be the circle group fixing z. V_{2k} splits into a (basis dependent) sum of 1-D weight spaces $-2k, 2k, 2$, fixed by S^1_z. For a rotation g, we can compute matrix elements by using a hermitian pairing on each tangent space (v, gv) and integrating. Let's do this for when v is the zero weight vector.

Then gv is a zero-weight vector for $gS^1_z g^{-1} = S_{gz}$, the subgroup fixing axis gz. These matrix elements allow for a change-of-basis.

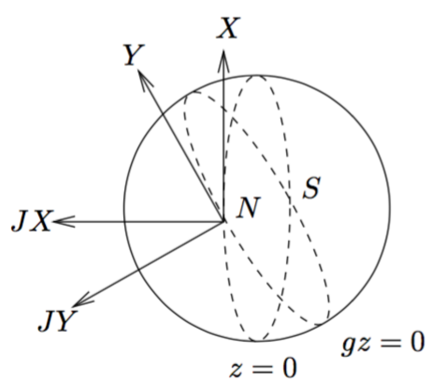
Pick a section s of $\mathcal{L}^{otimes2}$ that is S^1_z invariant and peaks at $z = 0$, then so does $s^{\otimes k}$. This is not normalized, but we can compute the Rayleigh quotient:

$$(v_0^{(k)}, gv_0^{(k)}) = \frac{(s^k, gs^k)}{(s^k, s^k)}$$

We've already done the denominator. Now for the numerator:

$$(s^k, gs^k) = \int_{S^2} \langle s^k, gs^k \rangle \, 2k\omega = k \int_{S^2} \langle s, gs \rangle^k \, 2\omega = k \int_{S^2} e^{k\psi} \; 2\omega$$

with $\psi = \langle s, gs \rangle$. Now we need to look for the stationary parts of ψ. We know when s and gs are maximized, individually. Together, they'll be maximized at these two antipodal points N and S:



Now $X \langle s, gs \rangle = \langle \nabla_X s, gs \rangle + \langle s, \nabla_X gs \rangle$. The first one has $\nabla_X s = -Xs + 2\pi i \mu(\langle s, gs \rangle)$ from the Konstant quantization formula and this is just $2\pi i \mu s$. The second term in gs can be manipulated if we write $X = pY + qJY$ since Y, JY span the tangent space. All together, this gives:

$$X\psi = 2\pi i \mu(\psi) - 2\pi i p \nu(\psi) - 2\pi q \nu(\psi)$$

And similarly since $Y = p'X + q'JX$ we get

$$Y\psi = -2\pi i \nu(\psi) + 2\pi i p' \mu(\psi) + 2\pi q' \mu(\psi)$$

Now we want the Hessian, so note $X\mu = 2\omega(X, X) = 0 = Y\nu$. On the other hand $X\nu = 2\omega(Y, X) = -Y\mu$.

Thus our matrix of second derivatives is:

$$-2\pi i(-2\omega(X,Y))\begin{pmatrix} e^{i\beta} & 1 \\ 1 & e^{i\beta} \end{pmatrix}$$

and that gives a corresponding Hessian determinant (the $\omega(X,Y)$ cancels when calculating everything):

$$\text{Hess}_N = -8\pi^2 i \sin\left(\beta e^{i\beta}\right)$$

where β is the angle. The Hessian at S is just the complex conjugate to this.

The last thing we need is $e^{\psi(N)}, e^{\psi(S)}$. For the latter, we get

$$e^{\psi(S)} = \langle s, gs\rangle (S) = \left\langle hh^{-1}s(hN), ghh^{-1}s(hN)\right\rangle = \langle hs(N), ghs(N)\rangle = \left\langle s(N), h^{-1}ghs(N)\right\rangle$$

This last bit is just a clockwise rotation at N's tangent space, which acts just like $e^{-i\beta}$. Similarly for N it would act as $e^{i\beta}$. All together this gives us our matrix element at long last:

$$(s^k, gs^k) = k\frac{2\pi}{k}\left(\frac{e^{-i\beta}}{\sqrt{8\pi^2 i\sin(\beta)e^{i\beta}}} + \frac{e^{i\beta}}{\sqrt{-8\pi^2 i\sin(\beta)e^{-i\beta}}}\right)$$

Dividing by $(s,s) \approx \sqrt{\pi k}$ we get the final result:

$$(v_0^{(k)}, gv_0^{(k)}) \approx \sqrt{\frac{2}{\pi k\sin\beta}}\cos\left((2k+1)\frac{\beta}{2} + \frac{\pi}{4}\right)$$

The modulus comes partly from the Hessian and the normalization factor in the previous subsection. The $\pi/4$ is a standard phase convention.

8 The Calculation for the $6j$ symbols

Going back, the $6j$ symbol was also an inner product of two $SU(2)$ invariant vectors in a 12-fold tensor product.

The irrep V_a can be identified with a sphere of radius a, with symplectic form given by $\omega_x(v,w) = \frac{1}{4\pi a^3}x.(v\times w)$. The complex structure $J_x(v) = \frac{1}{a}x\times v$. The metric is then naturally $B_x(v,w) = v.w$, just like the regular metric.

The group $SO(3)$ acts on the sphere, and μ can be obtained in the obvious way (explain this).

Tensor products of irreducibles can be viewed as line bundles over cartesian products of the spaces.

Now we form our manifold M like:

$$S_a^2 \times S_b^2 \times S_c^2 \times S_c^2 \times S_d^2 \times S_e^2 \times S_e^2 \times S_f^2 \times S_a^2 \times S_f^2 \times S_d^2 \times S_b^2$$

This lies in $(\mathbb{R}^3)^{12}$ and a vector on it can be denoted by $(x_1, \ldots, x_{12})$. $G = SO(3)$ acts on the sphere, preserving Kahler structure, and when V_a has a even, it acts on the corresponding line bundle as well.

Now G has three actions worth noting. Firstly, it can act diagonally on all 12 spheres, giving a moment map $\phi(x_1, \ldots, x_{12}) = \sum_i x_i$. This is algebraically the same as G acting diagonally on the tensor of the 12 irreps.

ALso, G can act by G^4 on the first set of three spheres, the second, etc, each copy acting diagonally on each set of 3 spheres.

The moment map is

$$\mu(x_1, \ldots, x_{12}) = (x_1 + x_2 + x_3, x_4 + x_5 + x_6, x_7 + x_8 + x_9, x_{10} + x_{11} + x_{12})$$

We have a well-defined invariant section $\tilde{s}_\mu = s^{abc} \otimes \cdots \otimes s^{fdb}$.

Lastly G^6 can act, each copy acting diagonally on pairs of V's of the same weight.

We have a well-defined invariant section $\tilde{s}_\nu = s^{aa} \otimes \cdots \otimes s^{ff}$.

From the very beginning, in the definition of the $6j$ symbols we get

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = (-)^{a+b+c+d+e+f} \frac{(\tilde{s}_\mu, \tilde{s}_\nu)}{||\tilde{s}_\mu|| ||\tilde{s}_\nu||} \prod_a (a+1)$$

This can be written into:

$$\begin{Bmatrix} ka & kb & kc \\ kd & ke & kf \end{Bmatrix} = (-)^{a+b+c+d+e+f} \frac{(\tilde{s}_\mu^k, \tilde{s}_\nu^k)}{||\tilde{s}_\mu^k|| ||\tilde{s}_\nu^k||} \prod_a (ka+1)$$

There are three integrals we need to do.

8.1 Integrals in the Denominator

Theorem 1. *Let $\tilde{s}_1, \tilde{s}_2$ be G-invariant sections of the line bundle $\mathcal{L}$ over M and let s_1, s_2 be the corresponding sections on the quotient L_G over M_G. Then*

$$(\tilde{s}_1, \tilde{s}_2) = \int_M \left\langle \tilde{s}_1^k, \tilde{s}_2^k \right\rangle k^n \Omega \approx \left(\frac{k}{2}\right)^{d/2} \int_{p \in M_G} \left\langle s_1^k, s_2^k \right\rangle \mathrm{Vol}(Gp) k^{n-d} \Omega_G$$

The proof requires some further knowledge of geometric invariant theory.

Applying this to M_{G^4}, M_{G^6} and noting that they are single-point spaces,

$$||\tilde{s}_\mu^k||^2 \approx \left(\frac{k}{2}\right)^6 \mathrm{Vol}(\mu^{-1}(0))$$

$$||\tilde{s}_\nu^k||^2 \approx \left(\frac{k}{2}\right)^6 \mathrm{Vol}(\nu^{-1}(0))$$

It remains to calculate $(\tilde{s}_\mu^k, \tilde{s}_\nu^k)$. As before, we care about the place where both moment maps are zero. That condition on ν requires all the x_i to sum to zero (6 are the negatives of the other six). The condition on μ forces each of those 6 to form a (euclidean) tetrahedron. If a euclidean tetrahedron can't be formed, then $\mu^{-1}(0) \cap \nu^{-1}(0) = \emptyset$ and we only have exponential decay of the $6j$ symbols, as desired. Otherwise there is a critical point.

The remainder of it requires using the stationary trick and computing the Hessian which is, in the authors words, "frustratingly difficult", so we just write it here for completeness

$$\begin{aligned}
\mathrm{Hess}_{[\tau]}(\psi) &= -i(2\pi)^{18} 2^9 e^{i\sum\theta_a} \left(\frac{[ace]}{4\pi}\right)^9 \\
&\quad \times \left(\frac{\mathrm{vol}(G^4\tau)}{\mathrm{vol}_\rho(G)^4}[ace]^4\right)^{-1} \left(\frac{\mathrm{vol}(G^6\tau)}{\mathrm{vol}_\rho(G/T)^6}\frac{[ace]^6}{\prod a}\right)^{-1} \left(\frac{[ace]}{\mathrm{vol}_\rho(G)}\right)^2 \\
&= -ie^{i\sum\theta_a}(2\pi)^9[ace]\left(\prod a\right)\left(\frac{\mathrm{vol}_\rho(G)^2\,\mathrm{vol}_\rho(G/T)^6}{\mathrm{vol}(G^4\tau)\,\mathrm{vol}(G^6\tau)}\right)
\end{aligned} \tag{12}$$

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Therefore

$$\psi([\tau'] = (-1)^{\sum a} e^{\frac{1}{2} ik \sum a\theta_a} \quad \text{and} \quad \psi([\tau]) = (-1)^{\sum a} e^{-\frac{1}{2} ik \sum a\theta_a}. \tag{13}$$

5.10 Putting it all together

We combine the original integral definition (7) with the asymptotic normalisation factors (8), (9), the reduction (10), the stationary phase evaluation (6) incorporating the Hessian (12) and 0–order terms (13).

$$\begin{Bmatrix} ka & kb & kc \\ kd & ke & kf \end{Bmatrix} \sim \left(\prod \sqrt{ka+1}\right) \left(\frac{k}{2}\right)^{-6} (\mathrm{vol}(\mu^{-1}(0))\,\mathrm{vol}(\nu^{-1}(0)))^{-\frac{1}{2}} k^9 \left(\frac{k}{2}\right)^{\frac{3}{2}} \times \left(\frac{2\pi}{k}\right)^9 \left\{ \frac{e^{-\frac{1}{2} ik \sum a\theta_a}}{\sqrt{-\operatorname{Hess}_{[\tau]}(\psi)}} + \frac{e^{\frac{1}{2} ik \sum a\theta_a}}{\sqrt{-\operatorname{Hess}_{[\tau']}(\psi)}} \right\}.$$

Substituting in the volumes $8\pi^2$ and 4π of G and G/T gives

$$\begin{Bmatrix} ka & kb & kc \\ kd & ke & kf \end{Bmatrix} \sim \sqrt{\frac{2}{3\pi k^3 V}} \cos\left\{ \sum (ka+1)\frac{\theta_a}{2} + \frac{\pi}{4} \right\}$$

where $V = \frac{1}{6}[ace]$ is the (scaling-independent) volume of τ. This completes the proof of the theorem.

9 Question

Can similar geometrically-meaningful formulae be obtained for general spin networks, the so-called $3nj$ symbols?