Graded Lie Algebras, Supersymmetry, and Applications

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Abstract. An introduction to graded Lie Algebras is given, with particular focus on the $\mathbb{Z}_2$-graded superalgebras. The Kac classification of graded Lie algebras is presented and their tensor representations are examined. The remainder of the paper is then devoted to their applications for studying dynamic symmetries of atomic nucliei.

1 Introduction

Ever since its introduction in the early 1970s, the concept of supersymmetry has spurred an immense amount of research both in pure mathematics and in theoretical physics. The first use of supersymmetry was in string theory, and it has since been considered a valuable tool for quantizing relativistic field theories, extending as far as quantum gravity. Beyond this purely theoretical purpose, it has also found applications in condensed matter physics, atomic physics, and nuclear physics. In the field of nuclear physics, exerimental evidence has demonstrated the existence of supersymmetry in certain nuclei.

2 Definitions and Key Concepts

2.1 Lie Algebras

A Lie algebra $\mathfrak{g}$ is defined as a vector space L over a field $\mathbb{F}$ equipped with a bilinear operation $[,] : L \times L \to L$ called the commutator bracket [6]. In this paper, the field shall be either $\mathbb{R}$ or $\mathbb{C}$ in which case the Lie algebra shall be called *real* or *complex*, respectively. The commutator satisfies the additional properties:

$$[A, B] = -[B, A] \tag{2.1}$$

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \tag{2.2}$$

Often, the Lie algebra will consist of linear transformations on an underlying vector space, in which case the commutator naturally is defined as

$$[A, B] := AB - BA \tag{2.3}$$

Now let X_α be a basis for the vector space L. Since $[X_\mu, X_\nu] \in L$, it can be expressed as a linear combination of the X_α, so that

$$[X_\mu, X_\nu] = \sum_\lambda c^\lambda_{\mu\nu} X_\lambda \tag{2.4}$$

the quantities $c^\lambda_{\mu\nu}$ are the structure constants of the Lie algebra.

Lie algebras appear in many areas of theoretical physics, ranging from the Poisson bracket $\{f, g\} := \sum_i (\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i})$ in classical mechanics and symplectic geometry to the celebrated Lie derivative $L_X(Y)$ on Riemannian manifolds used in general relativity theory. Perhaps most notable among these is the application of the commutator in the Heisenberg formulation of quantum mechanics, giving the equation of motion $\frac{dA}{dt} = \frac{i}{\hbar}[H, A] + \frac{\partial A}{\partial t}$.

We call a subspace M of L that satisfies $[M, M] \subseteq M$ a *Lie subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$. If M satisfies the stronger condition that $[L, M] \subseteq M$ then it is called an *ideal* of $\mathfrak{g}$. A Lie algebra that has no nontrivial ideals is called *simple*. If the Lie algebra can be decomposed as a direct sum $\mathfrak{g} = \bigoplus_i \mathfrak{h}_\mathfrak{i}$ of simple subalgebras h_i so that $[\mathfrak{h}_\mathfrak{i}, \mathfrak{h}_\mathfrak{j}] = 0$ for $i \neq j$ (h_i is an ideal), then we call $\mathfrak{g}$ *semisimple*.

At the end of the 19th century, the semisimple Lie algebras over $\mathbb{C}$ were completely classified, primarily due to the work of Cartan. All semisimple algebras are known to be expressible as a direct sum of simple algebras from the following list [6]:

Name	Cartan Label	Label in Physics	Matrix Realization
Special Linear	$A(n)$	$sl(n+1)$	$\mathrm{Tr}(X) = 0$
Special Orthogonal	$B(n)$	$so(2n+1)$	$X^T = -X$
Symplectic	$C(n)$	$sp(2n)$	$JX = -X^T J$
Special Orthogonal	$D(n)$	$so(2n)$	$X^T = -X$
Exceptional	G_2	G_2	
	F_2	F_2	
	E_6	E_6	
	E_7	E_7	
	E_8	E_8	

Where above, $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

The Lie algebras of particular importance shall be the real Lie algebras $u(n)$ and $su(n)$. $u(n)$ can be represented as the set of skew-Hermitian matrices so that $X^\dagger = -X$, and $su(n)$ is obtained by imposing tracelessness on these operators.

2.2 The Harmonic Oscillator Representation for Bosons and Fermions

Let b_α and $b_\alpha^\dagger, \alpha = \{1, \ldots, n\}$ be the creation and annihilation operators for bosons, each satisfying commutation rules akin to those of the quantum harmonic oscillator:

$$[b_\alpha, b_\beta^\dagger] = \delta_{\alpha\beta}, \qquad [b_\alpha, b_\beta] = [b_\alpha^\dagger, b_\beta^\dagger] = 0 \tag{2.5}$$

Now consider the fermion operators $a_i, a_i^\dagger, i \in \{1, \ldots, m\}$.

For the fermion operators, there are instead *anti*-commutation rules. These rely on the introduction of the anti-commutator, defined much like the commutator on operator algebras:

$$\{X, Y\} := XY + YX \tag{2.6}$$

With this, the anti-commutation rules are:

$$\{a_i, a_j^\dagger\} = \delta_{ij}, \qquad \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0 \tag{2.7}$$

We further have that the boson operators and the fermion operators commute

$$[b_\alpha, a_i] = [b_\alpha^\dagger, a_i] = [b_\alpha, a_i^\dagger] = [b_\alpha^\dagger, a_i^\dagger] = 0 \tag{2.8}$$

From this we can form the operator products:

$$B_{\alpha\beta} = b_\alpha^\dagger b_\beta, \qquad A_{ij} = a_i^\dagger a_j \tag{2.9}$$

As well as the mixed products

$$Y_{i\alpha}^{(+)} = a_i^\dagger b_\alpha, \qquad Y_{i\alpha}^{(-)} = b_\alpha^\dagger a_i \tag{2.10}$$

The first two are both boson operators while the second two are fermion operators. From the commutation relations of (2.5) and (2.7) we have that [3]

$$[B_{\alpha\beta}, B_{\gamma\delta}] = b_\alpha^\dagger b_\beta b_\gamma^\dagger b_\delta - b_\gamma^\dagger b_\delta b_\alpha^\dagger b_\beta = \delta_{\beta\gamma} B_{\alpha\delta} - \delta_{\alpha\delta} B_{\beta\gamma} \tag{2.11}$$

$$[A_{ij}, A_{kl}] = a_i^\dagger a_j a_k^\dagger a_l - a_k^\dagger a_l a_i^\dagger a_j = \delta_{jk} A_{il} - \delta_{il} A_{jk} \tag{2.12}$$

From this it is clear that we can form any element in the algebra from taking appropriate commutators:

$$[B, B] = B \tag{2.13}$$

$$[A, A] = A \tag{2.14}$$

From these relations, we see that the algebras generated by the B and A can be put into isomorphism with $u(n)$ and $u(m)$, respectively. We now have the additional anticommutation relations:

$$\{Y_{i\alpha}^{(+)}, Y_{\beta j}^{(-)}\} = a_i^\dagger b_\alpha b_\beta^\dagger a_j + b_\beta^\dagger a_j a_i^\dagger b_\alpha = \delta_{ij} B_{\beta\alpha} + \delta_{\beta\alpha} A_{ij} \tag{2.15}$$

So that $\{Y, Y\} \in A \cup B$. Lastly:

$$[Y_{i\alpha}^{(+)}, B_{\beta\gamma}] = a_i^\dagger b_\alpha b_\beta^\dagger b_\gamma - b_\beta^\dagger b_\gamma a_i^\dagger b_\alpha = \delta_{\alpha\beta} Y_{i\gamma}^{(+)} \tag{2.16}$$

Using this same logic it is easy to see in general that [3]

$$[Y_{i\alpha}^{(\pm)}, B_{jk}], [Y_{i\alpha}^{(\pm)}, A_{\beta\gamma}] \in Y^{(\pm)} \tag{2.17}$$

This lengthy, but elementary example shall later be referred to as the *graded Lie Superalgebra* $u(n/m)$.

2.3 Graded Lie Algebras

In general a $\mathbb{Z}$-graded Lie algebra is a Lie algebra where the underlying vector space of $\mathfrak{g}$ is a graded sum $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$, and we have $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$.

Motivated by this pervious example, we consider more generally what it means to introduce a $\mathbb{Z}_2$-*grading* on a Lie algebra. As we had before with the two distint sets of boson (B, A) and fermion $(Y^{(\pm)})$ generators, we consider a vector space over *two* sets of generators, denoted X_i and Y_i. A $\mathbb{Z}_2$-graded Lie algebra or *Lie superalgebra* $\mathfrak{g}^*$ is the vector space spanned by these generators and closed under commutation, along with anticommutation in the second family. We have the properties [1, 4]:

$$\begin{aligned}
[X_i, X_j] &= c_{ij}^k X_k \\
\{Y_i, Y_j\} &= f_{ij}^k X_k \\
[X_i, Y_j] &= d_{ij}^k Y_k
\end{aligned} \tag{2.18}$$

As before, the c_{ij}^k, f_{ij}^k and d_{ij}^k are called graded Lie structure constants. In the previous example, the X_i would represent the bosonic elements while the Y_i would represent the fermionic elements. For this reason, these elements are appropriately titled *bosonic* and *fermionic*, respectively.

Additionally, as with ordnary Lie algebras, there is an analogue of the Jacobi identity that must hold, [4]:

$$\begin{aligned}
[X_i,[X_j,X_k]]+[X_j,[X_k,X_i]]+[X_k,[X_i,X_j]] &= 0\\
[X_i,[X_j,Y_k]]+[X_j,[Y_k,X_i]]+[Y_k,[X_i,X_j]] &= 0\\
[X_i,\{Y_j,Y_k\}]+\{Y_j,[Y_k,X_i]\}-\{Y_k,[X_i,Y_j]\} &= 0\\
[Y_i,\{Y_j,Y_k\}]+[Y_j,\{Y_k,Y_i\}]+[Y_k,\{Y_i,Y_j\}] &= 0
\end{aligned} \tag{2.19}$$

We define a graded Lie subalgebra as a subspace of $\mathfrak{g}^*$ that is closed under commutation and anti-commutation [1] [2]. Additionally we define an ideal of a graded Lie algebra as a subspace of $\mathfrak{g}^*$ that is closed under commutation and anticommutation with all elements of $\mathfrak{g}^*$. A simple graded Lie algebra is one with no nontrivial ideals.

Occsaionally, instead of definining the anticommutator as a distinct entity seperate from the commutator, it is useful to define a *Lie superbracket* on all elements X and Y by

$$[A,B]_S = -(-1)^{|A||B|}[B,A]_S \tag{2.20}$$

where $|A|$ denotes the *degree* of A, which is either 0 of 1 depending on whether A is in the bosonic or fermionic part of the algebra.

2.4 Classification of Graded Lie Algebras

The classification of complex finite dimensional simple graded Lie algebras was first done by by Kac [5].

Name	Kac Label	Label in Physics
Special Unitary	$A(n,m)$	$su(n+1/m+1)$
Orthosymplectic	$B(n,m)$	$osp((2n+1)/2m)$
Orthosymplectic	$D(n,m)$	$osp(2n/2m)$
Others	$C[n]$	
	$A[n]$	
	$P[n]$	
	$F[4]$	
	$G[3]$	
	$D[1,2,\alpha]$	

We can define $su(n/m)$ analogously as we have defined $u(n/m)$ previously in section (2.2), but with the extra constraint of tracelessness for the initial boson and fermion operators. Similarly, the orthosymplectic groups can be obtained by requiring that the individual bosonic/fermionic parts of the superalgebra preserve a requisite bilinear form [5]. The other types of Lie superalgebras find less application in physics and so are outside the scope of this paper.

As before, particular attention shall be paid to the non-simple superalgebra $u(n/m)$.

3 The Representation Theory of Graded Lie Algebras

3.1 Over Lie Algebras

A very important topic in the representation theory of Lie algebras is the subject of their *tensor representations* under the action of the symmetric group.

Consider the action of $u(n)$ on tensors of rank N. The irreducible representations are given by tensors of specific symmetry type under interchange of indices. To each such representation, there is a corresponding partition of N into $\lambda_1 + \ldots + \lambda_n$ where each λ_i is a non-negative integer and we require $\lambda_1 \geq \ldots \geq \lambda_n$. Graphically, this is often described by a Young tableau [7]. For example, in the case $\lambda_1 = 4, \lambda_2 = 3, \lambda_3 = 1, \lambda_4 = 1$, we get:

$$\begin{aligned} &\lambda_1 : \square\square\square\square \\ &\lambda_2 : \square\square\square \\ &\lambda_3 : \square \\ &\lambda_4 : \square \end{aligned} \tag{3.1}$$

Such a tableau would be denoted $[\lambda_1, \lambda_2, \lambda_3, \lambda_4] = [4, 3, 1, 1]$. The totally symmetric case $[N]$ is represented by the following Young tableau:

$$\begin{aligned} &\lambda_1 : \underbrace{\square \ldots \square}_{n} \\ &\lambda_i = 0, \forall i > 1 \end{aligned} \tag{3.2}$$

Whereas the totally anti-symmetric case would be the vertical arrangement, $\lambda_i = 1, \forall i$.

3.2 Return to the Oscillator Representation: N-particle states

We now consider generalizations to *Young supertableaux*. The case that we are most interested in is $u(n/m)$. It is worth studying this algebra to get an understanding of how the operators $B_{\alpha\beta}$ and A_{ij} interact. We follow the method in [3]. As is standard, the raising and lowering operators act on the vacuum state $|0\rangle$. The one-particle states are $b_\alpha^\dagger |0\rangle$ and $a_i^\dagger |0\rangle$. The N-particle states are then given by:

$$\underbrace{b_{\alpha_1}^\dagger ... b_{\alpha_k}^\dagger}_{k} \underbrace{a_{i_1}^\dagger ... a_{i_{N-k}}^\dagger}_{N-k} |0\rangle \tag{3.3}$$

From this, the number operator is just:

$$\hat{N} = \hat{N}_B + \hat{N}_F = \sum_\alpha b_\alpha^\dagger b_\alpha + \sum_i a_i^\dagger a_i \tag{3.4}$$

More compactly, we can write this as

$$\hat{N} = \sum_a \zeta_a^\dagger \zeta_a, \tag{3.5}$$

Where we have defined the ζ_a to be $\zeta_a = b_\alpha$ for $a = \{1, \ldots, k\}$ and $\zeta_a = a_{i-k}$ for $a = \{k+1, \ldots, N\}$. These shall be called the *super*-creation and annihilation operators.

An N-particle state is symmetric in the first k boson indices and antisymmetric in the last $N-k$ fermion indices. As a young tableau, it is then:

$$\left(\overbrace{\square \ldots \square}^{k}, \quad \begin{matrix} \square \\ \vdots_{N-k} \\ \square \end{matrix} \right) \tag{3.6}$$

Such a state is then a direct product of a symmetric $u(n)$ representation and an antisymmetric $u(m)$ representation. In general, the space of all N-particle states inside of $u(n) \otimes u(m)$ is

$$(\overbrace{\square \ldots \square}^{N}, 0) \oplus (\overbrace{\square \ldots \square}^{N-1}, \square) \oplus \left(\overbrace{\square \ldots \square}^{N-2}, \begin{matrix} \square \\ \square \end{matrix} \right) \oplus \ldots \oplus \left(0, \begin{matrix} \square \\ \vdots_{N} \\ \square \end{matrix} \right) \tag{3.7}$$

The fermionic operators $S^{(+)}, S^{(-)}$ act as ladder operators on this space. Because of this the N-particle states $\boxtimes \dots \boxtimes$ are irreducible subrepresentations of $u(n/m)$.

3.3 Tableaux over Lie Superalgebras

The N-particle states above shall form a subrepresentation that will be labelled by the Young supertableaux:

$$\lambda_1 = N : \underbrace{\boxtimes\boxtimes \dots \boxtimes\boxtimes}_{N}$$
$$\lambda_i = 0, \forall i > 1$$

This shall be called the totally supersymmetric representation [1]. It is the representation in which all bosonic degrees of freedom are completely symmetric, and all the fermionic degrees of freedom are completely antisymmetric. This space is of chief importance in application to the study of nuclei. For information on the more general case of supertableaux, consult [3].

In general, as before, when studying the tensor representation of $u(n/m)$, we pick a set of positive integers $[\lambda_1, \dots, \lambda_n]$ whose sum shall be the tensor rank. In addition to not allowing for more than n rows this time, we also do not allow any of the λ_i to exceed m, so we cannot have more than m columns.

3.4 Branching

The maximal Lie algebra for $u(n/m)$ is the direct sum of the two bosonic $B_{\alpha\beta}$ and A_{ij}, $u(n) \oplus u(m)$. We have seen before how a totally supersymmetric state branches, and indeed in general the branching of $u(n/m)$ into $u(n) \oplus u(m)$ is similar to the classical $u(n+m)$ to $u(n) \oplus u(m)$ [1]. In the latter case:

$$\overbrace{\square \dots \square}^{N} = (\overbrace{\square \dots \square}^{N}, 0) \oplus (\overbrace{\square \dots \square}^{N-1}, 1) \oplus \dots \oplus (0, \overbrace{\square \dots \square}^{N}) \tag{3.8}$$

and the former case of $\boxtimes \dots \boxtimes$ is illustrated in section (3.2). The branching is

$$[N] \to (N, 0) \oplus (N-1, 1) \oplus \dots \oplus (0, N) \tag{3.9}$$

The total number of bosons plus fermions is the same in each part of the reduction.

4 Casimir Operators

4.1 Over Lie Algebras

For an ordinary Lie algebra we have the standard relation:

$$[X_i, X_j] = c_{ij}^k X_k \tag{4.1}$$

There is then a natural metric tensor on this vector space given by:

$$g_{ij} = c_{ik}^l c_{jl}^k \tag{4.2}$$

where we have now made use of Einstein summation convention. For semisimple Lie algebras, this is nonsingular, and has an inverse [6] that we write as

$$g^{ij} = (g^{-1})_{ij} \tag{4.3}$$

The invariant quadratic operator:

$$C_2 = g^{ij} X_i X_j \tag{4.4}$$

is easily seen to commute with all X_k, and thus makes it *Casimir*. A well-known example is the Casimir operator for total angular momentum $\mathbf{J^2} = \sum_i J_i^2$ in the angular momentum algebra $so(3)$.

In general, an operator C in the universal enveloping algebra of a Lie algebra is called a Casimir operator if

$$[C, X_k] = 0, \qquad \forall X_k \in \mathfrak{g} \tag{4.5}$$

A Casimir operator of degree d is one that can be written as a sum [7]:

$$C_d = \sum_{a_1,\ldots,a_d} c_{a_1,\ldots,a_d} X_{a_1} \ldots X_{a_d} \tag{4.6}$$

4.2 Over Lie Superalgebras

The notion of a Casimir operator over a Lie algebra can be generalized naturally to Lie superalgebras. We call an operator C superCasimir if it satisfies:

$$[C, X_a] = 0, \qquad [C, Y_b] = 0, \qquad \forall X_a, Y_b \in \mathfrak{g}^* \tag{4.7}$$

Over $u(n/m)$, the super-Casimir operators are known. Of particular importance is their action on the totally supersymmetric representation $[N]$. We have in fact already constructed one Casimir operator: the number operator $C_1 = \hat{N}$ whose eigenvalue is just N when acting on this representation. There is also a quadratic operator C_2 with eigenvalue $N(N+n-m-1)$, as demonstrated in [1].

5 Applications

5.1 Dynamic Symmetry and Spectrum Generating Algebras

A useful area of applications of Lie algebras involves the case when the Hamiltonian can be expressed as a functional of the elements of the algebra:

$$\hat{H} = f(X_1, \ldots, X_n), \qquad X_\alpha \in \mathfrak{g} \tag{5.1}$$

The Lie algebra is then called the *spectrum generating algebra* for this Hamiltonian [1, 2, 7]. Of particular interest is when the functional f is a polynomial in each of the X_i, so that we can write:

$$\hat{H} = E_0 + \sum_i a_i X_i + \sum_{i_1, i_2} a_{i_1,i_2} X_{i_1} X_{i_2} + ... \tag{5.2}$$

The special case of a *dynamic symmetry* occurs when the elements of the Hamiltonian are all Casimir operators for some chain of subalgebras $\mathfrak{g} \supset \mathfrak{g}' \supset \ldots$, then we can write:

$$\hat{H} = f(C_1, \ldots, C_m) \tag{5.3}$$

Because these are all Casimir operators for this chain of algebras, it is possible to diagonalize all of them simultaenously and obtain an *analytic* equation for the energies in terms of the Casimir eigenvalues (the quantum numbers).

5.2 Spectrum Generating Superalgeabras

The concept of a spectrum generating algebra can be generalized to a Hamiltonian that is written as a functional both of the bosonic and fermionic operators:

$$\hat{H} = f(X_1, \ldots X_n, Y_1, \ldots, Y_m) \tag{5.4}$$

Again the special case of when this Hamiltonian can be written in terms of superCasimir operators for a chain of superalgebras $\mathfrak{g}^* \supset \mathfrak{g}^{*'} \supset \ldots$ is called dyanmical *super*symmetry. Again, the superCasimir operators can be simultaneously diagonalized and give an analytic expression for all of the energies in terms of the quantum numbers. This is a powerful tool when studying systems of bosons and fermions.

5.3 Interacting Boson Model in the Atomic Nucleus

In this example of application, no distinction shall be made between protons and neutrons, and they shall instead both be called "nucleons". Individually, each nucleon is a fermion, but when they pair, each pair acts much like a boson. In this model, nucleons pair in either a $J = 0$ s-state or five $J = 2$ d-states, forming a six-dimensional representation of the group $U(6)$ [1, 2]. Although there is additional precision gained by not treating the proton and neutron pairs on equal footing, we shall ignore that correction in this paper. We label each type of bosonic pairing by an index $a \in \{1, \ldots, 6\}$. We then have that the Lie algebra is generated by

$$G_{\alpha\beta} = b^\dagger_\alpha b_\beta \tag{5.5}$$

Now $u(6)$ has three chains, corresponding to sets of Casimir operators that would give three different dynamic symmetries:

$$\begin{aligned}
&\text{(I) } u(6) \supset u(5) \supset so(5) \supset so(3) \supset so(2) \\
&\text{(II) } u(6) \supset su(3) \supset so(3) \supset so(2) \\
&\text{(III) } u(6) \supset so(6) \supset so(5) \supset so(3) \supset so(2)
\end{aligned} \tag{5.6}$$

The spectra of the nuclei are obtained by finding the eigenvalues of the hamiltonian acting on the totally symmetric basis [7] $[N] = \square\square \ldots \square$.

The Hamiltonian given by (I) can be written as a set of parameters $(E_0, \varepsilon, \alpha, \beta)$ multiplying all posibile combinations of Casimir operators of the algebra, to second order:

$$H^{(I)} = E_0 + \varepsilon C_1(u5) + \alpha C_2(u5) + \beta C_2(so5) + \gamma C_2(so3) \tag{5.7}$$

The Casimir operator for $so(2)$ is omitted becasue this will only make a difference if a magnetic field were present to split the orbital degeneracy. The energies are then directly found in terms of the quantum numbers $N, n_d, \nu, n_\Delta, L, M_L$ that label the irreps of the first chain. Of these, only n_d, ν, L will contribute directly:

$$E^{(I)} = E_0 + \varepsilon n_d + \alpha n_d(n_d + 4) + \beta\nu(\nu + 3) + \gamma L(L + 1) \tag{5.8}$$

This type of symmetry is experimentally observed in ^{110}Cd [1]. For the second dynamic symmetry's Hamiltonian, we have:

$$H^{(II)} = E_0 + \kappa C_2(su3) + \kappa' C_2(so(3)) \tag{5.9}$$

And the energies can be obtained in terms of the angular momentum quantum number of the $so(3)$ irreducible representation as well as the quantum labels (Elliott numbers) for the $su(3)$ representation.

$$E^{(II)} = E_0 + \kappa C_2(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu) + \kappa' C_2 L(L + 1) \tag{5.10}$$

This type of dynamic symmetry is observed in ^{156}Gd [1]. Lastly, the Hamiltonian of the third dynamic symmetry is:

$$H^{(III)} = E_0 + AC_2(so6) + BC_2(so(5)) + CC_2(so(3)) \tag{5.11}$$

With energies similar to the first:

$$E^{(III)} = E_0 + A\sigma(\sigma + 4) + B\tau(\tau + 3) + CL(L + 1) \tag{5.12}$$

This type of dynamic symmetry is observed in ^{196}Pt [1]. Note that all the examples had even-even nucliei, as is necessary in order to have complete pairing with no fermions left over. For examples of these energy spectra, consult [1, 2].

5.4 Dynamic Supersymmetry in the Case of Unpaired Nucleons

We now consider the case where, in addition to the paired bosons, there are unpaired sets of nucleons that act as fermions. We proceed as in [1]. The ladder operators are:

$$\begin{aligned} &b^\dagger_\alpha, \quad \alpha = \{1, \dots, 6\} \\ &a^\dagger_{j,m_j} = a^\dagger_i, \quad i = 1, \dots, k \end{aligned} \tag{5.13}$$

with the $b^\dagger_\alpha$ defined as before, and now the $a^\dagger_i$ representing the creation of a fermion with total spin j and z-component m_j. Our attention shall be when $j = 3/2$ is the only allowed total angular momentum. We then have $k = 2j + 1 = 4$.

The Hamiltonian can be written as

$$\hat{H} = \hat{H}_B + \hat{H}_F + \hat{H}_{BF} \tag{5.14}$$

with a bosonic, a fermionic, and an interaction term. Each term can be written out (as in [1]) as:

$$\begin{aligned}
\hat{H}_B &= E_{0,B} + \sum_{\alpha\beta} \varepsilon_{\alpha\beta} b^\dagger_\alpha b_\beta + \sum_{\alpha\alpha'\beta\beta'} u_{\alpha\alpha'\beta\beta'} b^\dagger_\alpha b^\dagger_{\alpha'} b_\beta b_{\beta'} \\
\hat{H}_F &= E_{0,F} + \sum_{ik} \eta_{ik} a^\dagger_i a_k + \sum_{ii'kk'} u_{ii'kk'} a^\dagger_i a^\dagger_{i'} a_k a_{k'} \\
\hat{H}_{BF} &= \sum_{\alpha\beta ik} w_{\alpha\beta ik} b^\dagger_\alpha b_\beta a^\dagger_i a_k
\end{aligned} \tag{5.15}$$

As before, we know that the algebra products $b^\dagger_\alpha b_\beta$, $a^\dagger_i a_k$, $b^\dagger_\alpha a_i$, $a^\dagger_i b_\alpha$ form the Lie superalgebra $u(n/m)$, in this case $u(6/4)$.

Now we wish to diagonalize the system's total Hamiltonian in the supersymmetric basis $[N\} = \boxtimes \boxtimes \ldots \boxtimes$.

Let us study the branching of the Lie superalgebra $su(6/4)$. One immediate way is by reducing it to its maximal Lie algebra, $su(6) \oplus su(4)$. The following possible branch patterns then occur:

$$
\begin{array}{ccccc}
 & & u(6/4) & & \\
 & \swarrow & & \searrow & \\
u^B(6) & & \oplus & & u^F(4) \\
\downarrow & & & & \downarrow \\
so^B(6) & & \oplus & & su^F(4) \\
 & \searrow & & \swarrow & \\
\downarrow & & spin(6) & & \downarrow \\
so^B(5) & & \downarrow & & sp^F(4) \\
 & \searrow & & \swarrow & \\
\downarrow & & spin(5) & & \downarrow \\
so^B(3) & & \downarrow & & su^F(2) \\
 & \searrow & & \swarrow & \\
 & & spin(3) & & \\
 & & \downarrow & & \\
 & & spin(2) & &
\end{array} \tag{5.16}
$$

Where the $spin(n)$ groups are isomorphic to the orthogonal $so(n)$ Lie groups but affort *spinor* representations in addition to the standard tensor representations. This means that the labels λ_i for the quantum numbers may take half-integer values.

States can now be characterized by irreducible labels in the following way:

$$\left| \begin{array}{ccccccccccccc} u(6/4) & \supset & u^B(6) & \oplus & u^F(4) & \supset & so^B(6) & \supset & spin(6) & \supset & spin(5) \supset spin(3) & \supset & spin(2) \\ N_{B+F} & & (N_B, & & N_F) & & \Sigma & & (\sigma_1,\sigma_2,\sigma_3) & & (\tau_1,\tau_2) \quad (v_\Delta, J) & & (M_J) \end{array} \right\rangle \tag{5.17}$$

The second possible branching is into an orthosymplectic algebra as follows:

$$\begin{array}{ccccc} & & u(6/4) & & \\ & & \downarrow & & \\ & & osp(6/4) & & \\ & & \downarrow & & \\ & & osp(5/4) & & \\ & \swarrow & & \searrow & \\ so^B(5) & & \oplus & & sp^F(4) \\ & \searrow & & \swarrow & \\ \downarrow & & spin(5) & & \downarrow \\ so^B(3) & & \downarrow & & su^F(2) \\ & \searrow & & \swarrow & \\ & & spin(3) & & \\ & & \downarrow & & \\ & & spin(2) & & \end{array} \tag{5.18}$$

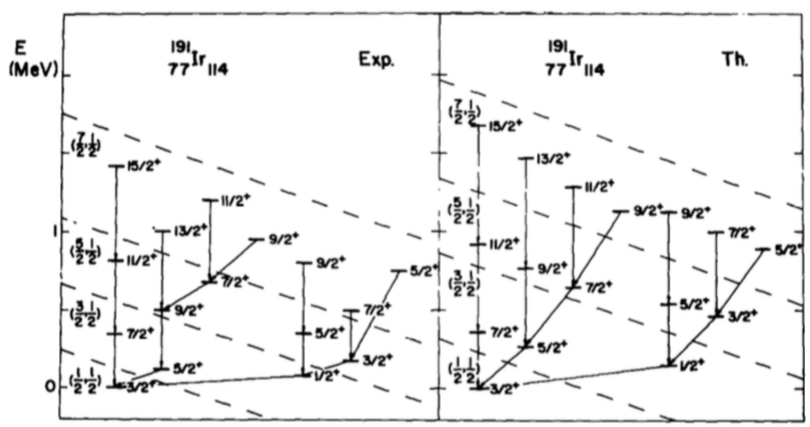
The first case of splitting shall be of principal interest to us. We know from section (4.2) that for Lie superalgebras $u(n/m)$ there are linear and quadratic Casimir operators. The rest follows from elementary knowledge of Casimir operators on ordinary Lie algeabras and can be found in references such as [7]. We can write a generic Hamiltonian for such a dynamic supersymmetry in terms of parameters $\eta, \eta', \beta, \gamma, e_i, i \in \{1, \dots 7\}$, scaling every possible combination of Casimir operators up to quadratic terms:

$$\begin{aligned} \hat{H} = & e_0 + e_6 C_1(u6/4) + e_7 C_2(u6/4) + e_1 C_1(u^B 6) + e_2 C_2(u^B 6) \\ & + e_3 C_1(u^F 4) + e_4 C_2(u^F 4) + e_5 C_1(u^B 6) C_1(u^F 4) \\ & + \eta C_2(o^B 6) + \eta' C_2(spin6) + \beta C_2(spin5) + \gamma C_2(spin2) \end{aligned} \tag{5.19}$$

giving energy eigenvalues of:

$$\begin{aligned} \hat{H} =& e_0 + e_6 N + e_7 N(N+1) + e_1 N_B + e_2 N_B(N_B+5) \\ &+ e_3 N_F + e_4 N_F(5-N_F) + e_5 N_B N_F + \eta\Sigma(\Sigma+4) \\ &+ \eta'(\sigma_1(\sigma_1+4) + \sigma_2(\sigma_2+2) + \sigma_3^2) \\ &+ \beta(\tau_1(\tau_1+3) + \tau_2(\tau_2+1)) + \gamma J(J+1) \end{aligned} \tag{5.20}$$

This energy formula gives a very definite branching structure to the energy levels of odd nuclei. It is seen to be qualitatively matched by experiment, and by adjusting the Hamiltonian's parameters properly, we can obtain a very close match to the experimentally observed splittings. Illustrated below is an example of this in ^{191}Ir, taken from [2]:



6 Conclusions

In this paper, we have introduced the mathematical formalism of Lie superalgebras and generalized many of the concepts from the theory of elementary Lie algebras to this broader setting. The theory of irreducible tensor representations has been developed and applied to describe the spectra of odd nuclei using dynamic supersymmetry. Moreover, experimental evidence has been presented. Currently, this is the only field with experimental validation of the phenomenon of supersymmetry [1]. For more detailed descriptions of the material , the reader is advised to consult with [2, 4]. For a more in-depth treatment of the supertableaux and the representation theory of superalgebras, the reader should consult [3].

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