# Chapter 1

$$(A+uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}$$

$$\Rightarrow \mathbb{1} = (A+uv^T)\left(A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}\right)$$

$$= \mathbb{1} + uv^TA^{-1} - \frac{uv^TA^{-1}}{1+v^TA^{-1}u} - \frac{uv^TA^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}$$

$$\Rightarrow (1+v^TA^{-1}u)\,uv^TA^{-1} - uv^TA^{-1} - uv^TA^{-1}uv^TA^{-1}$$

$$= v^TA^{-1}u \quad uv^TA^{-1} - (uv^TA^{-1})^2$$

$$(uv^TA^{-1}) \quad uv^TA^{-1} = v^TA^{-1}u \quad uv^TA^{-1}$$

$\propto u$ in both cases

$$u(v^TA^{-1}u)v^TA^{-1} = v^TA^{-1}u \quad uv^TA^{-1}$$

## Schur Complement

$$\underset{N\times N}{M} = \begin{pmatrix} M_{11\,(n\times n)} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \Rightarrow \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ -M_{22}^{-1}M_{21} & \mathbb{1} \end{pmatrix} = \begin{pmatrix} S & M_{12} \\ 0 & M_{22} \end{pmatrix}$$

$$S = M_{11} - M_{12}\,M_{22}^{-1}\,M_{21} = [M^{-1}]_{11}^{-1}$$

$$\begin{pmatrix} \mathbb{1} & -M_{12}M_{22}^{-1} \\ 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} S & M_{12} \\ 0 & M_{22} \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & M_{22} \end{pmatrix}$$

$$\Rightarrow M = \begin{pmatrix} \mathbb{1} & M_{12}M_{22}^{-1} \\ 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} S & 0 \\ 0 & M_{22} \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ M_{22}^{-1}M_{21} & \mathbb{1} \end{pmatrix}$$

$$\Rightarrow M^{-1} = \begin{pmatrix} \mathbb{1} & 0 \\ -M_{22}^{-1}M_{21} & \mathbb{1} \end{pmatrix}\begin{pmatrix} S^{-1} & 0 \\ 0 & M_{22}^{-1} \end{pmatrix}\begin{pmatrix} \mathbb{1} & -M_{12}M_{22}^{-1} \\ 0 & \mathbb{1} \end{pmatrix}$$

$$= \begin{pmatrix} S^{-1} & -S^{-1}M_{12}M_{22}^{-1} \\ -M_{22}^{-1}M_{21}S^{-1} & M_{22}^{-1} + M_{22}^{-1}M_{21}\,S^{-1}M_{12}\,M_{22}^{-1} \end{pmatrix}$$

When $M_{11}$ is $1\times1$

$$u = \begin{pmatrix} M_{11} \\ M_{21}/\sqrt{M_{11}} \end{pmatrix}$$

$$V' = \begin{pmatrix} M_{11} & M_{12} \end{pmatrix} \Big/ \sqrt{M_{11}}$$

$$\left(M_{22} - \frac{M_{21} M_{12}}{M_{11}}\right)^{-1} = M_{22}^{-1} + \frac{M_{22}^{-1} M_{21} M_{12} M_{22}^{-1}}{M_{11} - M_{12} M_{22}^{-1} M_{21}}$$

$$\begin{pmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{pmatrix} = \begin{pmatrix} \cancel{1} & 0 \\ 0 & M_{22} - \frac{M_{21} M_{12}}{M_{11}} \end{pmatrix} + uv^T$$

$$B = A^{-1}$$

$$G := \frac{\partial B_{kl}}{\partial A_{ij}} = \frac{1}{2}[A^{-1}]_{ik}[A^{-1}]_{jl} + \frac{1}{2}[A^{-1}]_{il}[A^{-1}]_{jk}$$

$\alpha$th eig of $A$

$$\Rightarrow \sum_{kl} G_{ij;kl} \left[\underbrace{v_{\alpha k} v_{\beta l}}_{(\lambda_\alpha \lambda_\beta)^{-1}} + \underbrace{v_{\alpha l} v_{\beta k}}_{(\lambda_\alpha \lambda_\beta)^{-1}}\right]$$

$$\Rightarrow \det_{sym} G = \prod_{\alpha, \beta \geq \alpha} \frac{1}{\lambda_\alpha \lambda_\beta} \Rightarrow \log \det_{sym} G = -\frac{1}{2} \sum_{\alpha,\beta} \log \lambda_\alpha \lambda_\beta - \frac{1}{2} \sum_\alpha \log \lambda_\alpha$$

$$= -(N+1) \log \det A$$

$\mathbb{R}^{\frac{N(N+1)}{2}}$ space of symmetric matrices

$$\Rightarrow \det G = (\det A)^{-N-1}$$

## Chapter 2

### 2.1 Normalized Trace

$$\tau(A) := \frac{1}{N}\mathbb{E}[\mathrm{Tr}\, A]$$

Often we look at $N \to \infty$ limit

$$\frac{1}{N}\mathrm{Tr}\, F(A) = \frac{1}{N}\sum_i F(\lambda_i) =$$

$$\langle F(\lambda)\rangle = \frac{1}{N}\sum_i F(\lambda_i)$$

Concentration: $\tau(F(A)) = \langle F(\lambda)\rangle_A$

When $\lambda_i \to \rho(\lambda)$

$$\Rightarrow \quad \tau(F(A)) = \int \rho(\lambda) F(\lambda) d\lambda$$

$m_k := \tau(A^k)$ $\qquad \|A\|_F^2 = m_2$

### 2.2 Wigner

$$X_{i\neq j} \sim N(0, \sigma_{od}^2)$$

$$X_{i=j} \sim N(0, \sigma_d^2)$$

$\tau(X) = 0$

$\tau(X^2) = \sigma_d^2 + (N-1)\sigma_{od}^2$

$\sigma_{od} = \sigma^2/N$

$\sigma_d = 2\sigma^2/N$ $\qquad \Rightarrow \tau(X^2) = \frac{\sigma^2}{2N}$

$\Rightarrow X = H + H^T \qquad H_{ij} \sim N(0, \frac{\sigma^2}{2N})$

$$\Rightarrow \tau(X^2) = 2\,\tau(H^2)$$
$$= \frac{2}{N}\cdot N^2 \cdot \frac{\sigma^2}{2N} = \sigma^2$$

$\tau(X^3) = 0$

$\tau(X^4) = 2\sigma^2$ as we will show $\Rightarrow$ non-Gaussian

In fact $\rho(\lambda) = \frac{\sqrt{4\sigma^2 - \lambda^2}}{2\pi\sigma^2}$

For $v \sim N(0, \sigma^2)$

$w = Ov$ is still $\sim N(0, \sigma^2)$

$\mathbb{E}[w_i w_j] = \sum O_{ik} O_{jl} \mathbb{E}[v_k v_l] = \sigma^2 O^T O = \sigma^2 \mathbb{1}$

Now $X = H + H^T$

$p(OH) = p(H) \quad \forall H$

$\Rightarrow p(OXO^T) = p(X)$

## 2.3 Resolvent

$$G_A(z) = \frac{1}{z\mathbb{1} - A}$$

Stieltjes:

$$g_N(z) = \tau(G_A(z)) = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{z - \lambda_k}$$

will write as $g_N$

Empirical spectral density

$$\rho_N(\lambda) = \frac{1}{N} \sum_k \delta(\lambda - \lambda_k)$$

$$g_N = \int_{-\infty}^{\infty} d\lambda \frac{\rho_N(\lambda)}{z - \lambda}$$

$g_N$ as $N \to \infty$

near $z \approx \infty$ as $N \to \infty$ $g(z) = \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} m_k(A)$

$\tau(A^k) = \lim_{N \to \infty} \frac{1}{N} \mathrm{Tr}\, A^k$

$$\tau(F(A)) = \int_{\lambda_-}^{\lambda_+} \rho(\lambda) F(\lambda) d\lambda$$

$\mathbb{E}_A[\rho_N(A)]$

Concretely we will use the fact that we only know $g(z)$ for large $z$

If $\min \lambda_l > c > 0$

$g(z)$ has a $z = \overset{\infty}{0}$ expansion

$$g(z) = -\sum_{k=0} z^k \tau(A^{-k-1})$$

$$g(0) = -\tau(A^{-1})$$

Ex 2.3.1

$$g_{\alpha A}(z) = \frac{1}{N}\sum_k \frac{1}{z-\alpha\lambda} = \frac{1}{\alpha} g(z/\alpha)$$

$$g_{A+\beta\mathbb{I}}(z) = \frac{1}{N}\sum_k \frac{1}{z-\beta-\lambda} = g(z-\beta)$$

## 2.3.2 Stieltjes for Wigner

Cavity method: Relate $g_N$ to $g_{N-1}$

and then equate as $N \to \infty$

$M = z - X$

$G_X = \frac{1}{z-X}$

$$\begin{array}{cc} 1\times 1 & \\ \downarrow & \\ \end{array}\begin{pmatrix} M_{11} & M_{1\ell} \\ M_{2\ell} & M_{22} \end{pmatrix} \quad M_{22} \in (N-1)\times(N-1)$$

$$\frac{1}{[G_X]_{11}} = M_{11} - \underbrace{\sum_{k,\ell} M_{1k} (M_{22})^{-1}_{k\ell} M_{\ell 1}}_{\text{concentrates at large } N}$$

$\Leftarrow$ Schur complement $S^{-1} = [M^{-1}]_{11}$

$$\mathbb{E}_X[M_{11}] = z$$

$$\mathbb{E}_{X_{1k}}\left[M_{1k} [M_{22}]^{-1}_{k\ell} M_{\ell 1}\right] = \frac{\sigma^2}{N} (M_{22})^{-1}_{k\ell}\, \delta_{k\ell}$$

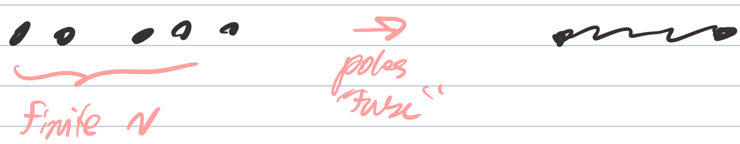
$\mathcal{N}(0, \sigma^2/N)_{N-1}$

$$\Rightarrow \mathbb{E}_{X_{1\cdot}}\left[\sum_{k,\ell} M_{1k} (M_{22}^{-1})_{k\ell} M_{\ell 1}\right] = \sigma^2 \underbrace{\tau(M_{22}^{-1})}_{\sigma^2 g}$$

$$\Rightarrow \frac{1}{g(z)} = z - \sigma^2 g(z) \Rightarrow g(z) = \frac{z}{2\sigma^2} \pm \frac{\sqrt{z^2-4\sigma^2}}{2\sigma^2}$$

Need this to be analytic for large $z \Rightarrow$ [illegible]
and $\to 0$

$$\Rightarrow \quad g(z) = \frac{z - z\sqrt{1 - 4\sigma^2/z^2}}{2\sigma^2}$$



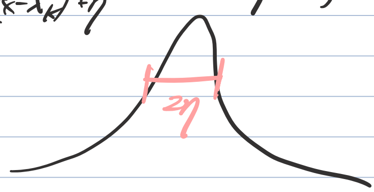

$$g(z) = \int d\lambda \, \frac{\rho(\lambda)}{z-\lambda}$$

$$g_N(x - i\eta) = \frac{1}{N}\sum_k \frac{x - \lambda_k + i\eta}{(x-\lambda_k)^2 + \eta^2}$$

real

$$\mathrm{Im}(g_N) = \frac{1}{N}\sum_k \frac{\eta}{(x-\lambda_k)^2+\eta^2} = (\pi K_\eta) * \rho$$

$$K_\eta(x) := \frac{1}{\pi}\frac{\eta}{x^2+\eta^2}$$




$\eta \ll N^{-1} \Rightarrow$ 0 to 1 eig in the interval $\Rightarrow g_N$ wildly fluctuates

$N^{-1} \ll \eta \ll 1$ (eg $\eta \sim N^{-1/2}$)

Then for $\Delta x \gg \eta$ $\rho$ is locally constant with

$$n \sim N\rho(x)\Delta x \Rightarrow N\eta \gg 1$$

$$\Rightarrow \frac{1}{N}\sum_{k:\lambda_k \in [x \pm \Delta x]} \frac{i\eta}{(x-\lambda_k)^2+\eta^2} \to i\int_{x-\Delta x}^{x+\Delta x} d\lambda \, \frac{\rho(\lambda)\,\eta}{(x-y)^2+\eta^2} \to i\pi\rho(x)$$

need $\Delta x \gg \eta$ for this

2.3.4

$$\lim_{\eta \to 0^+} \frac{1}{\pi}\,\mathrm{Im}\, g(x - i\eta) = \rho(x)$$

At finite size $N^{-1} \ll \eta \ll 1$. $\eta \sim N^{-1/2}$ works best

Want $\eta$ as small as possible to avoid blur

Error $\sim \eta \rho'(x)$

Want eta as large as possible for low statistical error

Error $\sim \frac{1}{N\eta}$

$\mathcal{E}_{tot} = \frac{1}{N\eta} + \eta \rho(\lambda) \Rightarrow \eta \sim \frac{1}{\sqrt{N}} \cdot \rho'(x)$

Density of Eigvals of a Wigner Matrix

$\rho(x) = \frac{1}{\pi} \lim_{\eta \to 0^+} \operatorname{Im} g(x - i\eta) = \frac{\sqrt{4\sigma^2 - x}}{2\pi\sigma^2}$ ⇐ semicircle law

$\Rightarrow$ Asymptotically no $\lambda > 2\sigma$, $\lambda < 2\sigma$

+ square root singularities

Will later show $\exists$ eigs in region of size $N^{-2/3}$ beyond thresh

2.3.3 Assume $A$ has $\tau(A^k \sim 1/k)$

a) $\Rightarrow g(z) \sim \sum_k \frac{1}{k} z^{-k-1}$

b) $-\frac{1}{z} \log(1 - 1/z)$

c) sing at $0 < z < 1$

d) $-\frac{1}{x - i\eta} \underbrace{\log\left(1 - \frac{1}{x - i\eta}\right)}$

$\frac{1}{\pi} \operatorname{Im}\left(\log\left(1 - \frac{1}{x} - \underbrace{i\eta}_{\to i\pi}\right)\right) \Rightarrow \frac{1}{x} \Rightarrow \rho(\lambda) = \frac{1}{\lambda}$

e) $\int_0^1 \frac{1}{\lambda} \frac{1}{z - \lambda} d\lambda = -\frac{1}{z}\left[\log(\epsilon - \epsilon z) - \log(\epsilon - z)\right] = \underbrace{-\frac{\log \epsilon}{z}}_{\text{div}} - \overbrace{\frac{1}{z} \log\left(\frac{z-1}{z}\right)}^{\text{as before}}$

f) $\sum \frac{1}{z^{2k+1}} = \frac{z}{z^2 - 1} \Rightarrow$ singular at $\pm 1$

$k=0$

$$1-i\eta \Rightarrow \frac{1-i\eta}{1-2i\eta-1} \Rightarrow \frac{1}{2\eta} \quad \Rightarrow \pm\frac{1}{2} \text{ mass}$$

$$-1-i\eta \Rightarrow \frac{1+i\eta}{1+2i\eta-1} \Rightarrow \frac{1}{2\eta}$$

(consistent with odd=0 even=1)

$$\int d\lambda \, \frac{\frac{1}{2}\delta(\lambda\pm1)}{z-\lambda} = \frac{1}{2}\left[\frac{1}{z-1} + \frac{1}{z+1}\right] = \frac{z}{z^2-1} \checkmark$$

## 2.3.6 Stieltjes transform on $\mathbb{R}$

$g_N \to \infty$ as $z \to \lambda_i$

as $N\to\infty$ $z$ is close to resolvent pole

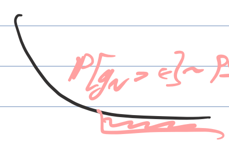
$d_i := |z-\lambda_i|$

$\mathbb{P}[d_i < \epsilon/N] = 2\epsilon\rho(z)$

$g_N(z) \approx \pm\frac{1}{N}\frac{1}{d_i}$ as $N\to\infty$ ← nearest one

$\mathbb{P}[|g_N| > \epsilon^{-1}] = \mathbb{P}[d_i < \epsilon/N] = 2\epsilon\rho(z)$

$\Rightarrow$ $g_N$ decays as $\frac{\rho(z)}{g^2}$



$\mathbb{P}[g_N > \epsilon] \sim \frac{\rho(z)}{\epsilon}$

$\rho(g_N) \sim \frac{\rho(z)}{g_N^2}$ for $g_N$ big

Nontrivial Claim $G_{11}$ is distributed like $\frac{1}{N}\mathrm{Tr}\, G$

$(z - A_N)^{-1}$

$$[G_X]_{11} \sim \frac{1}{M_{11} - \underbrace{\sum_{k,l} M_{1k}[M_{22}]^{-1}_{kl} M_{l1}}_{\Rightarrow \sigma^2 g'}}$$

$g' \sim P^{(N-1)}$

$$\Rightarrow P^{(N)}(g) = \int_{-\infty}^{\infty} dg' \, P^{(N-1)}(g')\,\delta\!\left(g - \frac{1}{z-\sigma^2 g'}\right)$$

$$g = \frac{1}{z-\sigma^2 g'}$$

This functional iteration gives rise to the fixed point:

$$P^{\infty}(g) = \frac{\rho(z)}{\left(g - \frac{z}{2\sigma^2}\right)^2 + \pi^2\rho(z)}$$

as $g\to\infty$ get $\frac{\rho(z)}{g^2}$

$y$

If we just had that $\lambda_i \underset{iid}{\sim} \frac{1}{\lambda^2}$ we'd expect Cauchy

The fact that we get it even w/ correlated $\lambda$s is super-universality

## Chapter 3

GOE GUE GSE

Because only $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ have division ring property

$\Rightarrow$ only 3 possible Gaussian Ensembles

Same $N \to \infty$ behavior

Correlations & large-$N$ deviations differ only through their dependence on the Dyson index $\beta$

$A = A^\dagger \Rightarrow \quad A = U \Lambda U^\dagger$

$z = x_r + i x_i \qquad x_r, x_i \sim N(0, \sigma^2/2)$

$\Rightarrow \mathbb{E}[|z|^2] = \sigma^2$

$\vec{x}$ a vector of complex centered Gaussians

$y = Ux$ is Gaussian w same var

Take $H$ a matrix of complex centered Gaussians

$X = H + H^\dagger$

$UXU^\dagger \sim X$

Want $\tau(X^2) = \sigma^2 + \mathcal{O}(1/N)$

$\Rightarrow \mathbb{E}[|H_{ij}|^2] = \frac{\sigma^2}{2N}$

$X_{ii}$ real w/ variance $\mathbb{E}[X_{ii}^2] = 1/N$

$X_{ij}$ complex w/ total var $\mathbb{E}[|X_{ij}|^2] = 1/N$

ie real & im parts have var $\frac{1}{2N}$

$$P[\{X_{ij}\}] = \exp\left[-\frac{N}{2\sigma^2} \operatorname{Tr} X^2\right]$$

For $\mathbb{R}, \mathbb{C}, \mathbb{H}$:

$$P[\{X_{ij}\}] = \exp\left[-\frac{N\beta}{4\sigma^2} \operatorname{Tr} X^2\right]$$

$\beta = 1, 2, 4$ resp.

Results of prior chapter apply

$\Rightarrow \rho(\lambda) = \frac{\sqrt{4\sigma^2 - \lambda^2}}{2\pi\sigma^2}$

## 3.1.2 Quaternionic Matrices

$i^2 = j^2 = k^2 = ijk = -1$ $\quad ij = -ji = k$ etc

$h = x_r + i x_i + j x_j + k x_k$ $\quad 1^* = 1 \quad i^* = -i$ etc

$|h|^2 = h h^* = x_r^2 + x_i^2 + x_j^2 + x_k^2$

$A = A^\dagger$ =: Hermitian quaternionic

$S S^\dagger = 1$ =: Symplectic

Using Pauli matrices $\quad i = i\sigma_z \quad j = i\sigma_y \quad k = i\sigma_x$

imaginary unit

can think of $A$ as $\underbrace{2N \times 2N}_{=: Q(A)}$

$$Q(\mathbb{1} \cdot j) := \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

$\Rightarrow$ $\dagger$ acts just like hermitian conjugation

Need 2 properties for Hermitian Quaternionic:

1) $Q^\dagger = Q$

2) $Q^R := Z Q^T Z^{-1} = Q^\dagger = Q$

In this $2N \times 2N$ rep'n symplectic matrices have

$$S S^\dagger = S S^R = \mathbb{1}$$

Off-diag $\mathbb{E}[|X_{ij}|^2] = 1/N$ $\quad \Rightarrow$ each part of $X_{ij}$ has var $\frac{1}{4N}$

diag are real $\mathbb{E}[X_{ii}^2] = 1/N \leftarrow \frac{1}{2N}$ ?

$\Rightarrow P[\{X_{ij}\}] \propto \exp\left[-\frac{N}{\sigma^2} \operatorname{Tr} X^2\right]$

$\mathbb{V}[|X_{ij}|^2] = \frac{2}{\beta}$ by LLN

$\Rightarrow \beta \to \infty$ would imply $|X_{ij}|^2 = 1$

Ex 3.1.1 1) $Z = j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow Z^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = Z^T$

$\Rightarrow Z Z^T Z^{-1} = Z^\dagger$

2) For $\mathbb{1}, \sigma_x, \sigma_y, \sigma_z$ $\quad ZQ^T Z^{-1} = Q^\dagger$

3) By linearity so does the $\mathbb{R}$-span

4) $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ does not satisfy this

## 3.1.3 Ginibre Ensemble

$H_{ij}$ indep from $H_{ji}$ $\Rightarrow$ eigs uniform on $\mathbb{D}(0, \sigma^2) \subset \mathbb{C}$

$\mathbb{E}[H_{ij} H_{ji}] = \rho\sigma^2 \Rightarrow$ ellipse $(1+\rho)\sigma$ along real $(1-\rho)\sigma$ along imag

$H^T H$ Wishardt $\to$ $\sqrt{s_i}$ has quarter circle (next chapter)

Ex 3.1.2

## 3.2 Moments

$\tau(X^4) = \frac{1}{N} \sum_{ijkl} \mathbb{E}[X_{ij} X_{jk} X_{kl} X_{li}]$

For nonzero : 1) All 4 are equal
2) They are equal pairwise

1) $\Rightarrow \frac{3\sigma^4}{N^2} \cdot N^2 \cdot \frac{1}{N} \to 0$

2) 3 ways

i) $X_{ij} = X_{jk} \qquad X_{kl} = X_{li} \qquad j \neq l$

$$\Rightarrow \frac{1}{N} \sum_{i,j\neq l} \mathbb{E}[X_{ij}^2 X_{il}^2] = \frac{1}{N} \cdot N \cdot N(N-1) \left(\frac{\sigma^2}{N}\right)^2$$

$$\to \sigma^4$$

ii) $X_{ij} = X_{li} \qquad X_{jk} = X_{kl} \quad \Rightarrow \sigma^4$

iii) $X_{ij} = X_{kl} \qquad X_{jk} = X_{li} \quad \Rightarrow \quad i \neq k$

$\Rightarrow i=l \quad j=k \qquad \Rightarrow i=j \quad k=l \quad \to i=k \Leftarrow$

$\Rightarrow \quad \tau(X^4) = 2\sigma^4$

Can show $\tau(X^{2n}) = C_k \, \sigma^{2k}$

$\tau(X^{2n+1}) = 0$

$$C_k = \sum_{j=0}^{k-1} C_j \, C_{k-j-1} \qquad C_0 = C_1 = 1$$

$$= \binom{2k}{k} \frac{1}{k+1}$$

3.22 Catalan

$\tau(X^{2k})$:

$a_1 a_2 \quad a_2 a_3 \quad a_3 a_4 \quad \dots \quad a_{2k} a_1$

$1 \quad 2 \quad 3 \quad \dots \quad 2k$

$\Rightarrow$ Only need to look @ 2-point pairings $\alpha_i = a_j$

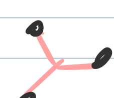
since $\alpha_i = \alpha_j = \alpha_k = \alpha_l$ will be sub-leading in $n$

$\Rightarrow (2k-1)!!$ pairings

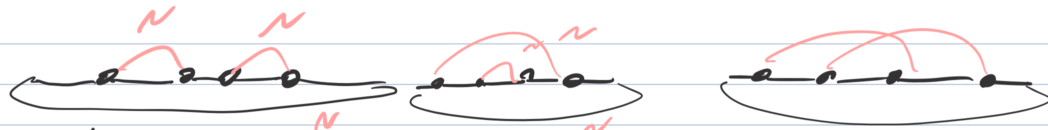
Crossing vs non-crossing:

— $= \mathbb{E}$

— $=$ matmul



is

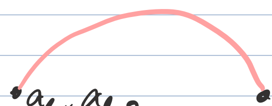



$O(1/N)$

Prove by induction $N$ non-crossing $N$ contribute
$k=1,2$ taken care of

Always $\exists$ a pair of consecutive points that are joined

Double line




$\alpha_\ell \alpha_{\ell+1}$ $\alpha_{\ell+1} \alpha_{\ell+2}$ $\alpha_{\ell+2}\, \alpha_{\ell+3}$ $\alpha_{\ell+2}\, \alpha_{\ell+3}$

take $\alpha_{\ell+1} = \alpha_{\ell+3}$ (any other option has all $\alpha$ equal $\Rightarrow$ no $N$ factor)

$\Rightarrow \alpha_{\ell+2}$ free $\Rightarrow$ factor of $N$ + deletion

$\alpha_{\ell+1} = \alpha_{\ell+3} \Rightarrow \alpha_\ell = \alpha_{\ell+1}$ now also $= \alpha_{\ell+3}$

$\Rightarrow$ left w/ $\sigma^2 \cdot \underbrace{\text{a partition of } 2k-2}_{\sim \sigma^{2k-2} \text{ by induction}}$

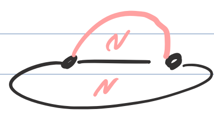
The other index match $\alpha_{\ell+1} = \alpha_{\ell+2} = \alpha_{\ell+3}$ is a subset of the first

$\sigma(X^{2k})$ involves $2k$ matrices, $4k$ indices w/ Tr and matmul
forcing $2k$ to be equal

$\frac{1}{N}\mathrm{Tr}$ + $k$ var terms $\rightarrow \dfrac{\sigma^{2k}}{N^{k+1}}$

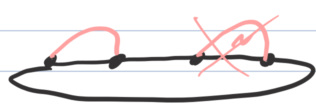
$\Rightarrow$ Need $k+1$ free indices

Eg $k=1$




$\Rightarrow \frac{1}{N^2} \sum \mathrm{Tr}[X_{i_1 i_2} X_{i_2 i_1}]$

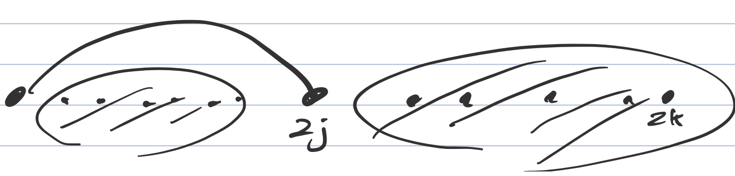
$k=2$




$+ \ldots$

For crossing, we match indices that are not equal
a priori $\Rightarrow$

$C_k = \sum_j$



$$= \sum_{j} C_{j-1} \, C_{k-j}$$

$$g(z) = \sum_{k=0}^{\infty} \frac{C_k \sigma^{2k}}{z^{2k+1}}$$

$$\Rightarrow g(z) - \frac{1}{z} = \sum_{k=1}^{\infty} \frac{\sigma^{2k}}{z^{2k+1}} \sum_{j=0}^{k-1} C_j \, C_{k-j-1}$$

$$= \frac{\sigma^2}{z} \sum_{j=0}^{\infty} \frac{C_j \, \sigma^{2j}}{z^{2j+1}} \sum_{k=j+1}^{\infty} \frac{C_{k-j-1}}{z^{2(k-j-1)+1}} \sigma^{2k-2j-2}$$

2 shift

$$= \frac{\sigma^2}{z} \sum_{j=0}^{\infty} \frac{C_j \sigma^{2j}}{z^{2j+1}} \sum_{k=0}^{\infty} \frac{C_k}{z^{2k+1}} \sigma^{2k}$$

$$= \frac{\sigma^2}{z} g(z)^2$$

$$\Rightarrow g(z) = \frac{z - z\sqrt{1 - 4\sigma^2/z^2}}{2\sigma^2}$$

Using $\sqrt{1-x} = 1 - \sum_{k=0}^{\infty} \frac{2}{k+1} \binom{2k}{k} \left(\frac{x}{4}\right)^{k+1}$

we get $\dfrac{z - z\sqrt{1 - \frac{4\sigma^2}{z}}}{2\sigma^2} = \dfrac{z}{\sigma^2} \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{2k}{k} \left(\frac{\sigma^2}{z^2}\right)^{k+1}$

$$= \sum_{k=0}^{\infty} \frac{C_k}{z^{2k+1}} \sigma^{2k}$$

## Chapter 4: Wishardt

$N \times T$ $\frown$

\# dims  \# samples

$$E_{ij} := \frac{1}{T} \sum_t x_i^t x_j^t \in \mathbb{R}^{N \times N}$$

sample cov

$$E = \frac{1}{T} H H^T$$ $H$ is $N \times T$ data matrix

$H_{it} = x_i^t$

$E$ sym psd

$\Rightarrow \lambda_k^E \geq 0$

$$F = \frac{1}{N} H^T H \in \mathbb{R}^{T \times T}$$

observation covariance

$q := N/T$

$\lambda^F = q^{-1} \lambda^E$

$q \leq 1 \Rightarrow F$ has $N$ nonzero $\lambda$

$T-N$ $\lambda = 0$

$$g_T^F(z) = \frac{1}{T} \sum_k \frac{1}{z - \lambda_k^F} = \frac{T-N}{T} \frac{1}{z} + \frac{1}{T} \sum_k \frac{1}{z - q^{-1} \lambda_k^E}$$

$$= \frac{1-q}{z} + q^2 g_E(qz)$$

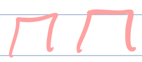
### 3.1.2

$$\mathbb{E}[H_{it} H_{js}] = C_{ij} \delta_{ts}$$

$$\Rightarrow \mathbb{E}[E_{ij}] = \frac{1}{T} \sum_t \mathbb{E}[H_{it} H_{jt}] = C_{ij} \Rightarrow \tau(E) = \tau(C)$$

$$\tau(E^2) = \frac{1}{N} \frac{1}{T^2} \mathbb{E} \operatorname{Tr}[H H^T H H^T]$$

$$= \frac{1}{NT^2} \sum_{ijts} E[H_{it} H_{jt} H_{js} H_{is}]$$

 $= \tau(C^2)$

 $= \frac{1}{NT^2} \sum_{t,i,j} C_{ii} C_{jj} = \frac{N}{T} \tau(C)^2 = q\, \tau(C)^2$

 $= \frac{1}{NT^2} \sum_{t,i,j} (C_{ij})^2 = \frac{1}{T} \tau(C^2) \to 0$

asymptotic limit

$$\Rightarrow \tau(E^2) = \tau(C^2) + q\, \tau(C)^2$$

$$C = \alpha \mathbb{I} \Rightarrow \tau(C^2) - \tau(C)^2 = 0$$
$$\tau(E^2) - \tau(E)^2 = q\, \alpha^2$$

## 4.1.3 Law of Wishart Matrices

$$P(\{H\}) \propto \exp\left[-\tfrac{1}{2} \mathrm{Tr}(H^T C^{-1} H)\right]$$
$$\propto \exp\left[-\tfrac{T}{2} \mathrm{Tr}(E C^{-1})\right]$$

$$H \to E = \frac{HH^T}{T}$$

$$\mathrm{Jac}^{-1}(E) = \int dH\, \delta\left(E - \frac{HH^T}{T}\right)$$
$$= \int dH\, d\hat{H} \exp\left\{ i \mathrm{Tr}\left(\hat{H}E - \hat{H}\frac{HH^T}{T}\right)\right\}$$

integrate out $H$ now

wlog assume $E$ diagonal (can always rotate frame $H \to OH$)

$$= \int d\hat{H} \exp\left\{ i \mathrm{Tr}\, \hat{H}E - \tfrac{T}{2} \log \det \hat{H} \right\}$$

$$\hat{H} \to E^{-1/2} \hat{H} E^{-1/2}$$

$$d\hat{H} = \prod_i d\hat{H}_{ii} \prod_{j>i} d\hat{H}_{ij} \to \prod_i E_{ii}^{-1} \underbrace{\prod_{j>i} (E_{ii}E_{jj})^{-1/2}}_{(\det E)^{-\frac{N-1}{2}}} d\hat{H} = (\det E)^{-1} \prod_{j\neq i} (E_{ii}E_{jj})^{-1/4} d\hat{H} = (\det E)^{-1} (\det E)^{-\frac{N-1}{2}} d\hat{H}$$

$$= \int d\hat{H} \exp\left\{ i \mathrm{Tr}\, \hat{H} - \tfrac{T}{2} \log \det \hat{H} + \tfrac{T}{2} \log \det E - \tfrac{N+1}{2} \log \det E \right\}$$

$(\det E)^{-\frac{N+1}{2}} (\det E)^{\frac{T}{2}}$

$\propto (\det E)^{\cdot} (\det E)^{-}$

$$\Rightarrow P(E) = \underbrace{\frac{(\det E)^{\frac{T-N-1}{2}}}{(\det C)^{T/2}} \exp\left[-\frac{T}{2} Tr[EC^{-1}]\right]}_{\text{Wishart}} \times \frac{(T/2)^{NT/2}}{\underbrace{\Gamma_N(T/2)}_{\text{multivar } \Gamma \text{ fn}}}$$

generalizes $\Gamma$-distribution

$$P(E) \propto \frac{1}{(\det C)^{T/2}} \exp\left[-\frac{T}{2} Tr[EC^{-1}] + \frac{T-N-1}{2} Tr \log E\right]$$

$-\frac{N}{2}(1-q^{-1})$

$C = 1$ $E \to W$ as $N, T \to \infty$

$$P(W) \propto \exp\left[-\frac{N}{2} Tr\, V(W)\right]$$

$$V(W) = q^{-1} \overset{C=1}{W} + (1-q^{-1}) \log W$$

rotationally inv. when $C=1$

## 4.2 Cavity derivation of Marčenko-Pasteur

### 4.2.1

$$g_W(z) = \tau(G_W(z)) \qquad G_W(z) = \frac{1}{z-W} = M^{-1} \qquad M = z\mathbb{1} - \frac{1}{T} HH^T$$

$$\frac{1}{G_{11}} = M_{11} - M_{12} M_{22}^{-1} M_{21}$$ Schur

off-diag $W$

$$= z - W_{11} - \frac{1}{T^2} \sum_{t,s}^{T} \sum_{j,k=2}^{N} H_{1t} H_{jt} [M_{22}^{-1}]_{jk} H_{ks} H_{1s}$$

$N(0,1)$ $N(0,1)$

$$\Omega_{ts} = \frac{1}{T} H_{\cdot t}^T M_{22}^{-1} H_{\cdot s}$$

$$= z - W_{11} - \frac{1}{T^2} \sum_{jk} Tr\left[H_{j\cdot} [M_{22}^{-1}]_{jk} H_{k\cdot}\right] + O(T^{-1/2})$$

$O(\gamma T^{-1/2})$

$\gamma^2 = \frac{1}{T} Tr[\Omega^2]$

$$= z - W_{11} - \frac{1}{T} \sum_{jk \geq 2} W_{kj} [M_{22}^{-1}]_{jk}$$

$$= z - W_{11} - \frac{1}{T} Tr[\underbrace{W_{22} G_{22}}] -$$

$1+O(T^{-1/2})$ $\mathrm{Tr}\left[W_2\,(z-W_2)^{-1}\right] = \sum_i \frac{\lambda_i}{z-\lambda_i} = \sum_i \left(\frac{z}{z-\lambda_i} - 1\right)$

$= -\mathrm{Tr}\,1 + z\,\mathrm{Tr}\,G_{22}$

$\Rightarrow \frac{1}{G_{11}(z)} = z-1+q-qzg + O(N^{-1/2})$ ⇐ ⇒ concentrates

$\frac{1}{G_{11}} = \frac{1}{E\,G_{11}} + O(N^{-1/2})$

$\mathbb{E}\,G_{11} = \frac{1}{N}\,\mathbb{E}\,\mathrm{Tr}\,G = g(z)$ by rotational invariance

$\Rightarrow \frac{1}{g(z)} = z-1+q-qzg(z)$

4.2.2

for asymptotic $1/z$

$\Rightarrow g(z) = \frac{z+q-1-\sqrt{(z+q-1)^2-4qz}}{2qz}$

$= \frac{z-(1-q)-\sqrt{z-\lambda_+}\sqrt{z-\lambda_-}}{2qz}$ $\qquad \lambda_\pm = (1\pm\sqrt{q})^2$

Finding correct branch is subtle!

$\Rightarrow \rho(x) = \frac{\sqrt{(\lambda_+-x)(x-\lambda_-)}}{2\pi qx}$ $\qquad \lambda_- < x < \lambda_+$

If $q>1$ ⇒ pole @ $z=0$ ⇒ $\underbrace{\frac{q-1}{q}}_{N-T\text{ trivial zero eigs}}\delta(x)$

$\rho_{1/q}(x) = q^2\rho(q\lambda)$

$\Rightarrow \rho(x) = \frac{\sqrt{[(\lambda_+-x)(x-\lambda_-)]_+}}{2\pi qx} + \frac{q-1}{q}\,\delta(x)\,\Theta(q-1)$

Ex 4.2.1

a) $(z+q-1)^2-4qz = z^2+2(q-1)z+(q-1)^2-4qz$

$$= z^2 - 2(q+1)z + (q-1)^2$$

$$\Rightarrow z = q+1 \pm \sqrt{(q+1)^2 - (q-1)^2}$$

$$= q+1 \pm 2\sqrt{q} = (1 \pm \sqrt{q})^2$$

b) $$\frac{z+q-1 - z\sqrt{1-\frac{\lambda_-}{z}}\sqrt{1-\frac{\lambda_+}{z}}}{2qz} \Rightarrow \underbrace{\frac{q-1}{2q}}_{\tau(X^0)}\frac{1}{z} + \underbrace{\frac{\lambda_- + \lambda_+}{4qz}}_{\frac{q+1}{2q}} = \frac{1}{z} : \tau(X^0) \checkmark \quad : O(z^{-1})$$

$O(z^{-2})$ $$\frac{1}{2q}\left(-\frac{\lambda_+\lambda_-}{4} + \frac{\lambda_-^2}{8} + \frac{\lambda_+^2}{8}\right) = 1 \Rightarrow \tau(X^1) = 1$$

$(q-1)^2$

c) Regular @ $z=0$ if $q<1$

$$\frac{z+q-1 - \sqrt{z-\lambda_-}\sqrt{z-\lambda_+}}{2qz} = \frac{z+q-1+}{2qz}$$

$z \to 0$

$$= \frac{z+q-1+\sqrt{\lambda_+\lambda_-}\sqrt{(1-\frac{z}{\lambda_+})(1-\frac{z}{\lambda_-})}}{2qz}$$

$q<1$ at $O(z^0)$ $\lambda_+\lambda_- = 1-q$ $$= \frac{1}{2q} + \frac{\lambda_+^{1/2}\lambda_-^{-1/2} + \lambda_-^{1/2}\lambda_+^{-1/2}}{2q} = \frac{1}{q} = \tau(X^{-1})$$

$$\frac{q-1+\sqrt{\lambda_+\lambda_-}}{2qz} = \frac{q-1+\sqrt{(1+\sqrt{q})(\sqrt{q}-1)}}{2qz} = \frac{q-1}{q}\cdot\frac{1}{z} \Rightarrow \delta(x)\cdot\frac{q-1}{q}$$

d) $$\int_{\lambda_-}^{\lambda_+} dx \frac{\sqrt{(x-\lambda_-)(\lambda_+-x)}}{2\pi q x} = \frac{(\sqrt{\lambda_+}-\sqrt{\lambda_-})^2}{4q} = \frac{\begin{cases} (1+\sqrt{q}-(1-\sqrt{q}))^2 & q<1 \\ (1+\sqrt{q}-(\sqrt{q}-1))^2 & q>0 \end{cases}}{4q} \Rightarrow \begin{cases} 1 & q<1 \\ 1/q & q>1 \end{cases}$$

$$\frac{1}{\pi q}\int_{\sqrt{\lambda_-}}^{\sqrt{\lambda_+}} \frac{ds}{s}\sqrt{(s^2-\lambda_-)(s^2-\lambda_+)} \Rightarrow$$

$\Rightarrow$ need $\delta(\lambda)\cdot\frac{q-1}{q}$

$s = \sqrt{x}$

$\Rightarrow ds = \frac{1}{2}\frac{1}{\sqrt{x}}dx \Rightarrow dx = 2s\,ds$

e) $q=1 \Rightarrow \lambda_- = 0 \quad \lambda_+ = 4$

$$\frac{1}{2\pi q}\int_0^4 dx \frac{\sqrt{x(4-x)}}{2\pi x} = \frac{1}{\pi q}\int_0^2 \underbrace{ds\sqrt{4-s^2}}_{\text{semicircle}}$$

$s = \sqrt{x}$

$$\frac{ds}{s} = \frac{1}{2}\frac{dx}{x}$$

f) . . .

## Chaper 5: Joint Distributions

$$P(M) \sim Z_N^{-1} \exp\left[-\frac{N}{2} \operatorname{Tr} V(M)\right]$$

matrix potential

$V(x) = \frac{x^2}{2\sigma^2}$ for Wigner

$V(x) = \frac{x + (q-1) \log x}{q}$ for Wishart

$V(x) = \frac{x^2}{2} + g \frac{x^4}{4}$ $\leftarrow$ Physics

### 5.1.2 Matrix Jacobian

$M$ (symm) has $\frac{N(N+1)}{2}$ vars

$M = O \Lambda O^T$ has $\Lambda \sim N$
$O \sim \frac{N(N-1)}{2}$

$M \to (\Lambda, O)$ introduces $\det(\Delta)$

$$\Delta := \left[\frac{\partial M}{\partial \Lambda}, \frac{\partial M}{\partial O}\right] \in \mathbb{R}^{\frac{N(N+1)}{2}, \frac{N(N+1)}{2}}$$

$[DM] = d^{N(N+1)/2}$ $\quad [D\Lambda] = d^N$ $\quad [DO] = d^0$

$$[|\det \Delta|] \sim d^{N(N-1)/2}$$

WLOG take $M$ diagonal $\qquad$ $M \to OMO^T$ has jac $= 1$

$O = \mathbb{1} + \epsilon \delta O$

$\delta O = -\delta O^T \Rightarrow \epsilon \delta O = \sum_{k<l} \theta_{kl} A^{(kl)}$

$[A^{(kl)}]_{kl} = 1 \quad [A_{kl}]_{lk} = -l$ else $0$

$M + \delta M = (\mathbb{1} + \theta_{kl} A^{(kl)})(\Lambda + \delta\Lambda)(\mathbb{1} - \theta_{kl} A^{(kl)})$

$\Rightarrow \delta M = \delta\Lambda + \theta_{kl}\left[A^{(kl)}\Lambda - \Lambda A^{kl}\right]$

$\Rightarrow \dfrac{\partial M_{ij}}{\partial \Lambda_{nn}} = \delta_{in}\delta_{jn}$

$\dfrac{\partial M_{ij}}{\partial \theta_{kl}} = \delta_{ik}\delta_{jl}(\lambda_l - \lambda_k)$

$k<l$ always!

$\Rightarrow \Lambda = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & \lambda_2-\lambda_1 & & \\ & & & & & \lambda_3-\lambda_1 & \\ & & & & & & \ddots \\ & & & & & & & \lambda_N-\lambda_{N-1} \end{pmatrix}$ first $N$

$\Rightarrow \det \Lambda = \prod_{k<l} (\lambda_l - \lambda_k)$

Vandermonde det

$\Rightarrow P(\{\lambda_i\}) \propto \prod_{k<l} (\lambda_k - \lambda_l) \exp\left[-\frac{N}{2}\sum_i V(\lambda_i)\right]$

not independent!

repulsion!

Ex 5.1.1

$O = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \Lambda = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$

$\Rightarrow O\Lambda O^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \lambda_1\cos\theta & -\lambda_1\sin\theta \\ \lambda_2\sin\theta & \lambda_2\cos\theta \end{pmatrix}$

$= \begin{matrix} \lambda_1\cos^2\theta + \lambda_2\sin^2\theta & (\lambda_2-\lambda_1)\cos\theta\sin\theta \\ (\lambda_2-\lambda_1)\cos\theta\sin\theta & \lambda_1\sin^2\theta + \lambda_2\cos^2\theta \end{matrix}$

$$\frac{\partial M}{\partial\{\lambda_1,\lambda_2,\theta\}} = \begin{matrix} \cos^2\theta & \sin^2\theta & 2(\lambda_2-\lambda_1)\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & 2(\lambda_1-\lambda_2)\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & (\lambda_2-\lambda_1)\ \cos^2\theta-\sin^2\theta \end{matrix}$$

$$|\det(\dots)| = |\lambda_2-\lambda_1| \quad :)$$

## 5.1.2

a) $|\lambda_2-\lambda_1| \exp\left[-\frac{\lambda_1^2}{2\sigma^2}-\frac{\lambda_2^2}{2\sigma^2}\right]$

b) $|\lambda_-| \exp\left[-\frac{\lambda_+^2+\lambda_-^2}{4\sigma^2}\right]$

$\rightsquigarrow |\lambda_-| \exp\left[-\frac{\lambda_-^2}{4\sigma^2}\right]$ take $x=|\lambda_-|>0$

c) $\Rightarrow P(x) = \frac{\pi}{2} x \exp\left(-\frac{\pi x^2}{4}\right)$ for $\mathbb{E}[x]=1$, $x>0$

d) $\beta=2$ affects Vandermonde det $\to x^\beta$

$\propto x^2 \exp\left[-\frac{x^2}{\sigma}\right] \to \frac{32}{\pi^2} x^2 \exp\left[-\frac{4}{\pi}x^2\right]$

## 5.2 Coulomb Gas

For $\beta$ ensemble

$$P(\{\lambda_i\}) = Z_{N\beta}^{-1} \prod_{k<l} |\lambda_l-\lambda_k|^\beta \exp\left[-\frac{\beta}{2}N\sum_i V(\lambda_i)\right]$$

$$= Z_{N\beta}^{-1} \exp\left[-\frac{\beta}{2}\left[\sum_i NV(\lambda_i) - \sum_{i\neq j}\log|\lambda_i-\lambda_j|\right]\right]$$

$T = \frac{2}{\beta}$ with $N$ particles in potential $NV(x)$

interacting w/ $V(x,y) = -\log|x-y|$

$\beta\to\infty$ has evals "Freeze"

Also for $N\to\infty$, $\beta$ fixed — log likelihood

$$P(\{\lambda_i\}) \propto \exp\left[\tfrac{1}{2}\beta N \mathcal{L}\right] \qquad \mathcal{L} = -\sum_i V(\lambda_i) + \frac{1}{N}\sum_{i\neq j}\log|\lambda_i-\lambda_j|$$

$$\frac{\partial L}{\partial \lambda_i} = 0 \Rightarrow V'(\lambda_i) = \frac{2}{N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}$$

Multiply both sides by $\frac{1}{z-\lambda_i}$ and sum

$$\Rightarrow \frac{1}{N} \sum_i \frac{V'(\lambda_i)}{z-\lambda_i} = \frac{2}{N^2} \sum_{i \neq j} \frac{1}{\lambda_i - \lambda_j} \frac{1}{z-\lambda_i}$$

$$= \frac{1}{N^2} \sum_{i \neq j} \frac{1}{(z-\lambda_i)} \frac{1}{(z-\lambda_j)} = g_N^2(z) - \frac{1}{N^2} \sum_i \frac{1}{(z-\lambda_i)^2}$$

$$= V'(z) g_N(z) - \underbrace{\frac{1}{N} \sum_i \frac{V'(z) - V'(\lambda_i)}{z - \lambda_i}} \qquad = g_N^2(z) - \frac{1}{N} g_N'(z)$$

$$= V'(z) g_N(z) - \Pi_N(z)$$

If $V$ is poly of degree $k$
$\Pi_N$ is also poly of degree $k-1$

Ex 5.2.1

a) $\lambda_1, \lambda_2, \lambda_3$ - expect $\lambda_3 = -\lambda_1$ $\quad \lambda_2 = 0$

b) $V'(\lambda_2) = \frac{2}{3} \left( \frac{1}{\lambda_2 - \lambda_1} + \frac{1}{\lambda_2 + \lambda_1} \right)$ $\qquad \lambda_2 = 0 \Rightarrow V'(0) = 0$ ✓

$$\Rightarrow V'(\lambda_1) = \frac{2}{3} \left( \frac{1}{\lambda_1} + \frac{1}{2\lambda_1} \right) = \frac{1}{\lambda_1} \quad \Rightarrow \lambda_1 = \pm 1$$

c) $g_3(z) = \frac{z^2 - 1/3}{z^3 - z}$ ✓

d) $\Rightarrow \Pi_3(z) = 1$

e) $z g_3(z) - 1 = \frac{2}{3} \frac{1}{z^2 - 1} = g(z)^2 - \frac{1}{N} g'(z)$

5.2.3 As $N \to \infty$ $g$ is self-averaging

so $\langle g_N \rangle = (g_{MLE})_{\text{mode}}$

$$V'(z)\,g(z) - \Pi(z) = g^2(z)$$

$$\Rightarrow g(z) = \frac{V'(z)}{2} \pm \frac{\sqrt{V'(z)^2 - 4\Pi(z)}}{2}$$ ← $\rho(x) > 0 \Rightarrow$ discriminant $< 0$

for $x$ w $\rho(x) \neq 0$

$$\operatorname{Re} g(x) = ⨍ \frac{\rho(\lambda)\,d\lambda}{x-\lambda} = \frac{V'(z)}{2}$$ "Hilbert Transform"

Inverse question: given $\rho$ does there exist a generalized orthogonal ens (or $\beta$ ens) with that $\beta$

if Hilbert transform exists, then yes

Ex 5.2.2

a) $g(z) = -\log \frac{z-1}{z}$ $\quad z \in (0,1)$

$= -i\pi - \underbrace{\log \frac{1-z}{z}}_{V'(z)/2}$

b) $⨍_0^1 \frac{d\lambda}{x-\lambda} = -\log|x-1| + \log x$

$= \log \frac{x}{1-x}$

c) $\Rightarrow V(x) = 2\underbrace{[x \log x + (1-x)\log(1-x)]}_{4 D_{KL}(x \,\|\, 1/2)} + C$

5.3 Applications

$V(z) = \frac{z^2}{2\sigma^2} \Rightarrow V'(z) = \frac{z}{\sigma^2} \Rightarrow \Pi(z) = \frac{1}{\sigma^2}$

$\Rightarrow g(z) = \frac{z - \sqrt{z^2 - 4\sigma^2}}{2\sigma^2}$

$V'(z) = \frac{1}{q}\left(1 + \frac{q-1}{z}\right)$

$\Rightarrow zV'(z)$ is poly of degree 1 $\Rightarrow z\Pi(z)$ of degree 0

$\Rightarrow \Pi(z) = \frac{c}{z}$

$\Rightarrow \frac{z+q-1 - \sqrt{(z+q-1)^2 - 4cq^2 z}}{2qz}$

as $z \to \infty$ $\quad \frac{cq}{z} + O(z^{-2})$

$g(z) \sim z^{-1} \Rightarrow c = q^{-1}$

5.3.2

Assume limiting $\rho(\lambda)$ has no gaps

(ie $g(z)$ has only one cut)

Expect this if $V$ is convex

$\Rightarrow g(z) = \int_{\lambda_-}^{\lambda_+} \frac{\rho(\lambda)}{z-\lambda}\, d\lambda$

$\Rightarrow g$ is singular near $\lambda_\pm$

$g(x)$ has imaginary part only when $x \in (\lambda_-, \lambda_+)$

analytic elsewhere

$D := V'(z)^2 - 4\Pi < 0$ there

$D$ poly of degree $2k$

only even zeros away from $\lambda_\pm$

$\Rightarrow D = Q(z)^2 (z-\lambda_+)(z-\lambda_-)$ $\qquad Q$ poly of deg $k-1$

$\Rightarrow g(z) = \frac{V'(z) \pm Q(z)\sqrt{(z-\lambda_+)(z-\lambda_-)}}{2}$

By enforcing $g(z) \sim \frac{1}{z}$ as $z \to \infty$

get $k+2$ constraints

$\Rightarrow$ get all $k+1$ coeffs of $Q$ and $\lambda_\pm$

$$\Rightarrow \rho(\lambda) = \frac{Q(\lambda)\sqrt{(\lambda_+-\lambda)(\lambda-\lambda_-)}}{2\pi}$$

Near edge $\rho(\lambda_\pm \mp \delta) \sim \sqrt{\delta}$ unless $Q$ has root of order $n$ there $\rightarrow \delta^{n+1/2}$

$\rightarrow$ gives rise to $N^{-2/3}$ effects (Ch 14) $N^{-2/(3+2n)}$

## 5.3.3 $M^2 + M^4$

$$V(\lambda) = \frac{x^2}{2} + \frac{\gamma x^4}{4}$$

Symmetry $\Rightarrow \lambda_+ = -\lambda_- =: 2a$

$V' = z + \gamma z^3$ $\qquad$ $Q(z) = a_0 + a_1 z + \gamma z^2$

$$\Rightarrow g(z) = \frac{z + \gamma z^3 - (a_0 + \cancel{a_1 z}^{\,0} + \gamma z^2)\, z\sqrt{1 - \frac{4a^2}{z^2}}}{2}$$

$z^3$ coeff $\Rightarrow a_2 = \gamma$
$z^2$ coeff $\Rightarrow a_1 = 0$
$z$ coeff $\Rightarrow 1 - a_0 + 2\gamma a^2 = 0$
$z^{-1}$ coeff $\Rightarrow 2a^4\gamma + 2a^2 a_0 = 2$

$$\Rightarrow g(z) = \frac{z + \gamma z^3 - (1 + 2\gamma a^2 + \gamma z^2)\, z\sqrt{1 - 4a^2/z^2}}{2}$$

$$3\gamma a^4 + a^2 - 1 = 0 \Rightarrow a^2 = \frac{\sqrt{1+12\gamma} - 1}{6\gamma}$$

$$\Rightarrow \rho(\lambda) = \frac{(1 + 2\gamma a^2 + \gamma\lambda^2)\sqrt{4a^2 - x^2}}{2\pi} \quad \text{for } \gamma > -\tfrac{1}{12}$$

At $\gamma = -1/12 \Rightarrow a = \sqrt{2}$ $\qquad$ $\rho(\lambda) = \dfrac{(8 - x^2)^{3/2}}{24\pi}$

$$g(z) = \frac{z^3}{24}\left(\left(1-\frac{8}{z^2}\right)^{3/2} - 1 + \frac{12}{z^2}\right)$$

## 5.4 Fluctuations at large $N$

$$\frac{\partial \mathcal{L}}{\partial \lambda_i} = \frac{2}{N}\sum_{j\neq i} \underbrace{\frac{1}{\lambda_i-\lambda_j}}_{O(N)} - \underbrace{V'(\lambda_i)}_{\sim O(1)}$$

Coulomb repulsion ⇒ force of order $d^{-1} \sim N$ on each eig

$d \sim 1/N$

$V'$ is of order $O(1)$ ⇒ repulsion dominates on small scales

⇒ equidistributed on small scales

$[\lambda - \frac{L}{2N}, \lambda + \frac{L}{2N}]$

$L \ll 1 \qquad \rho'(\lambda)\frac{L}{N} \ll \rho(\lambda)$

No fluctuations ⇒ $n(L) = \rho(\lambda)L + O(1)$

For Poisson process $n(L) = \rho(\lambda)L + \underbrace{\xi}_{\sim \mathcal{N}(0,1)}\sqrt{\rho(\lambda)L}$

We get $n(L) = \rho(\lambda)L + \sqrt{\Delta}\,\xi$ super small for large $n$

$\Delta := \frac{2}{\pi^2}\left[\log \tilde{n} + C\right] + O(\tilde{n}^{-1})$ ⇒ quasi-crystalline arrangement

Lets prove this:

$$H_{ij} = -\frac{\partial \mathcal{L}}{\partial \lambda_i \partial \lambda_j} = \begin{cases} V''(\lambda_i) + \frac{1}{N}\sum_{k\neq i}\frac{2}{(\lambda_i-\lambda_k)^2} & i=j \\ -\frac{2}{N}\frac{1}{(\lambda_i-\lambda_j)^2} & i\neq j \end{cases}$$

$\epsilon_i/N$ is the deviation of $\lambda_i$ from equilibrium

⇒ $P(\{\epsilon_i\}) \propto \exp\left[-\frac{\beta}{4N}\epsilon_i H_{ij}\epsilon_j\right]$ recall $P(\{\lambda_i\}) \propto e^{-\frac{1}{2}\beta N \mathcal{L}}$

⇒ $e^{-\frac{1}{4}\beta N \delta^T H \delta}$ $\quad \delta = \frac{\epsilon}{N}$

⇒ $\mathbb{E}\,\epsilon_i\epsilon_j = \frac{2N}{\beta} H^{-1}_{ij}$

locally $\lambda_i - \lambda_j \approx \frac{i-j}{N\rho}$

Take $V(x) = \frac{x^2}{2} \Rightarrow V'' = 1$

$$\Rightarrow H = \begin{pmatrix} 1+4N\rho^2+\frac{4N\rho^2}{4}+\cdots & -2N\rho^2 & \frac{-2N\rho^2}{4}\cdots \\ -2N\rho^2 & 1+4N\rho^2+\frac{4N\rho^2}{4}+\cdots & \\ -\frac{2N\rho^2}{4} & & \ddots \\ \vdots & & \end{pmatrix} \Big\} N$$

$$\hat{x}_q = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{-2\pi i k q/N} x_k \quad ;$$

translation acts as $e^{-2\pi i q/N}$

Up to bdy terms

$\Rightarrow \mu_q = 1 + 2N\rho^2$

$$\lambda \hat{x}_q = (1 + 2N\rho^2(2 - q - q^{-1} + \frac{2}{\ell^2} - \frac{q^2}{\ell^2} - \frac{q^{-2}}{\ell^2} - \cdots)$$

$$\Rightarrow \lambda = 1 + 4N\rho^2 \sum_{\ell} \frac{1-\cos(2\pi \ell q/N)}{\ell^2}$$

$du = \frac{2\pi q}{N}$ $\quad u = \frac{2\pi \ell q}{N} \Rightarrow \frac{1}{\ell^2} = \frac{(2\pi q)^2}{(Nu)^2}$

$\Rightarrow \frac{d\ell}{du}\frac{1}{\ell^2} = \frac{N}{2\pi q} \cdot \frac{(2\pi q)^2}{(Nu)^2} du$

$$\Rightarrow \mu_q = 1 + 4N\rho^2 \times \frac{2\pi}{N}|q| \int_0^\infty du \, \frac{1-\cos u}{u^2} = 1 + 4\pi^2\rho^2|q|$$

$\Rightarrow$ evals of $H^{-1} \sim \frac{1}{\mu_q}$

$n = i-j$ $\quad \frac{2\pi q}{N} = u$ $\quad \frac{dq}{du} = du\,\frac{N}{2\pi}$

$$\mathbb{E}[\epsilon_i \epsilon_j] = \frac{2N}{\beta} \frac{1}{N} \sum_q \frac{e^{2\pi i q n/N}}{1 + 4\pi^2\rho^2|q|} = \frac{2N}{\beta} \int_{-\pi}^{\pi} \frac{du}{2\pi} \frac{e^{iun}}{1 + 2\pi\rho^2|u|N} = \frac{2}{\beta} \int_{-\pi}^{\pi} \frac{du}{2\pi} \frac{e^{iun}}{N^{-1} + 2\pi\rho^2|u|}$$

$$\Rightarrow \mathbb{E}[(\epsilon_i - \epsilon_j)^2] = 2\mathbb{E}[(\epsilon_i)^2] - 2\mathbb{E}[\epsilon_i\epsilon_j] = \frac{4}{\beta} \int_{-\pi}^{\pi} \frac{du}{2\pi} \frac{1-\cos un}{N^{-1} + 2\pi\rho^2|u|}$$

$$= \frac{8}{\beta \cdot (2\pi\rho)^2} \int_0^{\pi} du \, \frac{1-\cos un}{u} \approx \frac{2}{\beta\pi^2\rho^2} \log n$$

$$d_{ij} = \frac{n}{N\rho} + \underbrace{\frac{\epsilon_i - \epsilon_j}{N}}$$

$\text{Var} \sim \frac{2}{\beta\pi^2\rho^2 N^2} \log n$

$$\Rightarrow \frac{n}{N\rho} \pm \frac{\xi}{N\rho}\sqrt{\Delta} \qquad \Delta \sim \frac{2}{\pi^2\beta} \log n$$

$$\frac{1}{N}\frac{n}{d_{ij}} = \frac{1}{N} \frac{n}{\frac{n}{N\rho} + \frac{\epsilon_i - \epsilon_j}{N}} = \frac{1}{\frac{1}{\rho} + \frac{\epsilon_i - \epsilon_j}{n}} \approx \rho + \rho^2 \frac{\epsilon_i - \epsilon_j}{n}$$

$$\Rightarrow \mathrm{Var}\, p = \frac{2p^2}{\pi^2\beta} \quad \frac{\log n}{n^2}$$

$$\Rightarrow \mathrm{Var}\, n = \mathrm{Var}\, pL = \frac{2}{\pi^2 \beta} \log \bar{n} \qquad \bar{n} = pL$$

## 5.4.2 Top eig deviations

$$P(\lambda_{max}; \{\lambda_i\}) = P_{N-1}(\{\lambda_i\}) \exp\left[-\frac{N\beta}{2}\left[V(\lambda_{max}) - \frac{2}{N}\sum_j^{N-1} \log \lambda_{max} - \lambda_j\right]\right)$$

The most likely positions of the $N-1$ $\lambda_j$ are changed by $O(\frac{1}{N^2})$ from their eq values under $P_{N-1}$

$$\frac{P(\lambda_{max}; \{\lambda_i\})}{P(\lambda_+; \{\lambda_i\})} \approx \exp\left[-\frac{N\beta}{2}\Phi(\lambda_{max})\right]$$ ← Large deviation principle

$$\Phi(x) = V(x) - V(\lambda_+) - \frac{2}{N}\sum_j^{N-1} \log\left(\frac{x - \lambda_j^*}{\lambda_+ - \lambda_j^*}\right)$$

$$\Phi'(x) = V'(x) - \frac{2}{N}\sum_j^{N-1} \frac{1}{x - \lambda_j^*} \rightarrow V'(x) - 2g(x)$$

$$\Rightarrow \Phi(x) = \int_{\lambda_+}^{x} (V'(s) - 2g(s))\, ds$$

$$= \int_{\lambda_+}^{x} \sqrt{V'(s)^2 - 4\Pi(s)}\, ds$$

$\Pi(s)$ of order $k-1$ $\Rightarrow$ for large $s$ $\Phi(x) \approx V(x)$
$(V')^2$ of order $2k$

(repulsion plays no role)

For $x - \lambda_+$ small but $\gg 1/N$ $\quad \sqrt{V'^2 - 4\Pi} \sim (s - \lambda_+)^\theta \cdot C$

$$\Phi(\lambda_{max}) \approx \frac{C}{\theta+1}(\lambda_{max} - \lambda_+)^{\theta+1}$$

recall $\pi\rho(\lambda) \simeq C(\lambda_+ - \lambda)^\theta$

$\Rightarrow \Phi(\lambda_{max}) \sim (\lambda_{max} - \lambda_+)^{3/2}$ generically

$\Rightarrow P(\lambda_{max}) \sim \exp\left(-\frac{2}{3}\beta\, C\, u^{3/2}\right) \qquad u = N^{2/3}(\lambda_{max} - \lambda_+)$

Ex 5.4.1

Wigner

$V(x) = \frac{x^2}{2} \rightarrow V' = x \qquad \Pi = 1 \qquad \lambda_+ = 2$

$\Rightarrow \int_2^x \sqrt{s^2 - 4}\, ds = \frac{1}{2} x\sqrt{x^2-4} - 2\log\left[\frac{\sqrt{x^2-4} + x}{2}\right]$

$\approx \frac{1}{2}x^2$ as $x \to \infty$

Wishart

$\sim 2\sqrt{x-2}$ as $x \to 2$

$V'(x) = 1 \qquad \Pi(x) = 1/x$

$\Rightarrow \int_2^x \sqrt{1 - \frac{4}{s}}\, ds = \sqrt{x(x-4)} + 2\log\left[\frac{x - \sqrt{(x-4)x} - 2}{2}\right]$

$\approx x$ as $x \to \infty$

$\sim 2\sqrt{x-4}$ as $x \to 4$

## 5.5 Eigval Density Saddle Point

$\omega(x) = \frac{1}{N}\sum_i \delta(\lambda_i - x)$ "density field"

$P(\{\omega\}) = Z^{-1}\exp\left\{-\frac{\beta N^2}{2}\left[\int dx\, \omega \cdot V - ⨍\, \omega(x)\,\omega(y) \log|x-y|\right] - N\int dx\, \omega(x) \log x\right\}$ ← ???

comes from change of vars often neglected since $N \to \infty$

$\frac{\delta}{\delta\omega} = 0 \Rightarrow V(x) = 2⨍ dy\, \omega^*(y) \log|x-y| + C$

$\Rightarrow V'(x) = 2⨍ dy \frac{\omega^*(y)}{x-y}$ } Tricomi type

Can solve using Tricomi-type equation ansatz:

$$f(x) = ⨍ dx' \frac{\rho(x')}{x-x'}$$

$$\Rightarrow \quad \rho(x) = -\frac{1}{\pi^2} \frac{1}{\sqrt{(x-a)(b-x)}} ⨍_a^b dx' \sqrt{(x'-a)(b-x')}\, \frac{f(x')}{x-x'} + C$$

1?

$$f(x) = \frac{V'(x)}{2} = \frac{x}{2} \Rightarrow -\frac{1}{\pi^2} \frac{1}{\sqrt{a^2-x^2}} \cancel{\frac{1}{2}} \int_{-a}^{a} dx' \sqrt{a^2-(x')^2}\, \frac{x'}{x-x'} = \frac{1}{\cancel{2}\pi^2} \frac{1}{\sqrt{a^2-x^2}} \left[ \frac{1}{2} a^2 \pi + x \underbrace{\int_{-a}^{a} dx' \frac{\sqrt{a^2-(x')^2}}{x'-x}}_{-\pi x^2} \right]$$

1?

$$= \frac{1}{2\pi} \sqrt{a^2-x^2} \quad \Rightarrow \quad a = 2 \quad \text{for normalization}$$

For $$V' = ⨍_0^\infty dy \frac{w^*(y)}{x-y} \Rightarrow \frac{1}{\pi^2} \frac{1}{\sqrt{x(a-x)}} \frac{1}{2} \int_0^a dx' \sqrt{x'(a-x')}\, \frac{x'}{x'-x}$$

$$= \frac{1}{2\pi^2} \frac{1}{\sqrt{x(a-x)}} \left[ \underbrace{\int_0^a dx' \sqrt{x'(a-x')}}_{\frac{1}{8}a^2\pi} + \underbrace{x \int_0^a dx' \frac{\sqrt{x'(a-x')}}{x'-x}}_{\frac{1}{2}x\pi(a-2x)} \right]$$

$$\to \frac{\pi}{2}(a+2x)(a-x)$$

$$= \frac{1}{4\pi} \sqrt{\frac{a-x}{x}}\, (a+2x)$$

$$a = \lambda_+ \quad \Rightarrow \quad w(x) = \frac{1}{4\pi} \sqrt{\frac{\lambda_+ - x}{x}}\, (\lambda_+ + 2x) \qquad \lambda_+ = \frac{4}{\sqrt{3}} \text{ for norm}$$

$$2 \to 4/\sqrt{3}$$

$$P(\text{matrix of all} > 0) \sim \exp[-\beta C N^2] \qquad C \sim \log 3/4$$

For $V = 0$ between two walls

$$w^* \sim \frac{1}{\pi} \frac{1}{\sqrt{(x-l_-)(x-l_+)}}$$

## Chapter 8: Addition of Random Variables & Brownian Motion

$X = X_1 + \dots + X_N$ has

$$P(x) = \int \prod [dx_i P(x_i)] \, \delta(x - \sum x_i)$$

$$\Rightarrow \quad \rho(k) = \rho_1(k) \cdots \rho_N(k)$$

$$\Rightarrow \quad M(k) = \log \rho(k) = \sum_i M_i(k)$$

"cumulants simply add"

In what follows we consider $dB$ gaussian

$$\mathbb{E}[dB] = 0 \qquad E[(dB)^2] = dt$$

### 8.2 Stochastic Calculus

Brownian Motion = Weiner process

$X_t$ is Gaussian of mean $\mu t$
and variance $\sigma^2 t$

Because of the infinite divisibility of Gaussians, can write:

$$X_{t_k} = \sum_{\ell=0}^{k-1} \mu \, \delta t + \sum_{\ell=0}^{k-1} \sigma \, \delta B_\ell \qquad t_k = \frac{kt}{N} \qquad \delta t = \frac{T}{N}$$

$\delta B_\ell \sim N(0, \delta t)$ for each $\ell$

As $N \to \infty$ $\quad \delta t \to dt$ $\quad \delta B_k \to dB$ $\qquad X_{t_N} = X_t$

$X_{t_k} \to$ continuous time Weiner

$$dX_t = \mu \, dt + \sigma \, dB_t \qquad X_0 = 0$$

$X_t$ depends on $X_{t'}$ but $X_{t'}$ & $X_t - X_{t'}$ are indep when $t' < t$

By convention $X_{t_k}$ does not depend on $dB_k$
$\Rightarrow X_t \perp dB_t$

$$\Rightarrow F(X(t+\delta t)) = F(X_t) + \delta X\, F'(X_t) + \frac{(\delta X)^2}{2} F''(X_t) + o(\delta t)$$

$$\delta X = \mu\, \delta t + \sigma\, \delta B$$

$$\Rightarrow (\delta X)^2 = \sigma^2 \delta t + \sigma^2 \underbrace{\left[(\delta B)^2 - \delta t\right]}_{\text{mean } 0} + o(\delta t)$$

$$\Rightarrow dF = \frac{\partial F}{\partial X} dX + \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial X^2} dt$$

More generally when $\mu(X_t, t)$ $\sigma(X_t, t)$ "general Itô process"

$$dX_t = \mu(X_t, t) dt + \sigma(X_t, t) dB_t$$

$$\Rightarrow dF_t = \frac{\partial F}{\partial X} dX + \left[\frac{\partial F}{\partial t} + \frac{\sigma^2(X_t, t)}{2} \frac{\partial^2 F}{\partial X^2}\right] dt$$

For $N$ indep $\{X_{i,t}\} = \vec{X}_t$

$$dX_{i,t} = \mu_i(\vec{X}_t, t) dt + \underbrace{dW_{i,t}}_{\text{mean } 0}$$

$$\mathbb{E}[dW_{i,t}\, dW_{j,t}] = C_{ij}(\vec{X}_t, t)\, dt$$

$$\Rightarrow dF_t = \frac{\partial F}{\partial X_i} \cdot dX_{i,t} + \left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial X_i \partial X_j} \frac{C_{ij}(\vec{X}_t, t)}{2}\right] dt$$

When noises are indep $C_{ij} \propto \mathbb{1} \Rightarrow$ only $\left(\frac{\partial}{\partial X_i}\right)^2 F$

## 8.2.3 Var as a function of $t$

Take $\mu = 0$ $\qquad F(X) = X^2$

$$\Rightarrow dF_t = 2X_t \underbrace{dX_t}_{\sigma\, dB} + \sigma^2 dt$$

$$\Rightarrow F_t = 2\sigma \int_0^t \underbrace{X_t\, dB}_{\text{indep}} + \sigma^2 t$$

$$\Rightarrow \mathbb{E}\, F_t = \sigma^2 t$$

## 8.2.4 Gaussian Addition

Use Itô's Lemma for $Z = X + \underbrace{Y}_{\text{Gaussian}}$

$Z \to Z_t$ brownian

$Z_0 = Y$

$$dZ_t = \mu dt + \sigma dB_t$$

$$F(Z_t) = \exp[ikZ_t]$$

$$dF_t = ik\, e^{ikZ_t} dZ_t - \frac{k^2\sigma^2}{2} F\, dt$$

$$= \left(ik\mu - \frac{k^2\sigma^2}{2}\right) F\, dt + ikF\sigma dB$$

$$\Rightarrow d\,\mathbb{E}[F_t] = \left(ik\mu - \frac{k^2\sigma^2}{2}\right) F\, dt$$

$$\Rightarrow \log \mathbb{E}[F_t] = \log \mathbb{E} F_0 + \left(ik\mu - \frac{k^2\sigma^2}{2}\right) t$$

take $t \to 1$

$\Rightarrow Z$ remains Gaussian

## 8.2.5 Langevin Equation

Goal: Construct a process s.t. steady state $X_t \sim P(X)$ for a given $P$

For $dX_t = dB_t$

Var $X_t$ grows unboundedly

Instead, rescale at each step:

$$X_{t+1} = \frac{X_t + dB_t}{\sqrt{1+dt}} \Rightarrow dX_t = dB_t - \frac{1}{2} X_t\, dt$$

"Ornstein Ullenbeck"

Converges to $N(0,1)$

Replace $-\frac{1}{2} X_t\, dt$ by $-\frac{V'(X_t)}{2} dt$

For $\mu\, dt$ we had $V = -\mu x$

$$\Rightarrow df = \frac{\partial f}{\partial X} dX + \frac{1}{2}\frac{\partial^2 f}{\partial X^2} \cancel{\sigma}\, dt$$

$$= f'(X_t)\left[dB_t - \frac{1}{2} V'(X_t)\, dt\right] + \frac{1}{2} f''(X_t)\, dt$$

Demanding $\frac{d(\mathbb{E} f)}{dt} = 0$

$$\Rightarrow 0 = -\mathbb{E}[f'(X_t)V'(X_t)] + \mathbb{E}[f''(X_t)]$$

$$\Rightarrow \mathbb{E}[f'(X_t)V'(X_t)] = \mathbb{E}[f''(X_t)]$$

$$\Rightarrow \forall f \quad \int dx\ p(x)\ f'(x)\ V'(x) = \int dx\ p(x)\ f''(x)$$

$$= \cancel{[p(x) f'(x)]_{-\infty}^{\infty}}^{\,0} - \int dx\ p'(x)\ f'(x)$$

$$\Rightarrow \forall f \quad \int dx\ p(x)\ f(x)\ V'(x) = -\int dx\ p'(x)\ f(x)$$

$$-p(x)V'(x) = p'(x)$$

$$\Rightarrow p(x) \propto \exp[-V(x)]$$

$\Rightarrow$ Given $p(x)$, define $V(x) = -\log P(x)$

$$dX_t = dB_t - \frac{1}{2}V'(X_t)\,dt$$

$$t \to \sigma^2 t \Rightarrow dX_t = \sigma\,dB_t - \frac{\sigma^2}{2}V'(X_t)\,dt$$

$V'(X_t)$ also acts like $\frac{\partial}{\partial X_t} \Rightarrow V'$ conjugate to $X$

For $N$ vars: $\quad dX_{i,t} = \sigma\,dB_i + \frac{\sigma^2}{2}\nabla_i \log P(\vec{x})\,dt$

Ex 8.2.1 Student's $t$

a) $P_\mu = Z_\mu^{-1}\left(1+\frac{x^2}{\mu}\right)^{-\frac{\mu+1}{2}}$

$$\Rightarrow V(x) = \frac{\mu+1}{2}\log 1+\frac{x^2}{\mu} + c$$

$$V'(x) = \frac{\mu+1}{2}\,\frac{2x/\mu}{1+x^2/\mu} = (\mu+1)\frac{x}{\mu+x^2}$$

b) $\mathbb{E}\left[\frac{x^2}{x^2+\mu}\right] = \mathbb{E}\left[\frac{xV'}{1+\mu}\right] = \frac{\mathbb{E}[1]}{1+\mu} = \frac{1}{1+\mu}$

c) $dX_t = dB_t - \frac{\mu+1}{\mu+x^2} x\,dt \qquad \mu\to\infty \Rightarrow dB - x\,dt$

Gaussian

d) Simulate

e) $V'(x) = \begin{cases} x \\ \frac{\mu+1}{\mu+x^2} x & \sim \frac{1}{x} \text{ as } x\to\infty \text{ (less restoring)} \\ & \sim (1+\frac{1}{\mu})x \text{ as } x\to 0 \text{ (stronger force)} \end{cases}$

## 8.2.6 Fokker-Planck

$P(x,t)$

$E[f'(X_t)dB_t]=0$

$$dE[f(X_t)] = E[f'(X_t)\underbrace{F(X_t)}_{-\frac{V'}{2}}]dt + \frac{\sigma^2}{2}E[f''(X_t)]\,dt$$

$$\Rightarrow \int f(x)\frac{\partial P(x,t)}{\partial t}dx = \int f'(x)F(x)P(x,t)dx + \frac{\sigma^2}{2}\int f''(x)P(x,t)dx$$

$$\underbrace{\Rightarrow \frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left(F(x)P(x,t)dx\right) + \frac{\sigma^2}{2}\frac{\partial^2 P(x,t)}{\partial x^2}}_{\text{Fokker-Planck}}$$

$$\frac{\partial P}{\partial t}=0 \Rightarrow P(x,t)\propto \exp\left[\frac{2F}{\sigma^2}\right] = \exp\left[-\frac{V'}{\sigma^2}\right]$$

# Chapter 9 Dyson Brownian Motion

## 9.1 Perturbation Theory

$$H = H_0 + \epsilon H_1$$

$$\lambda_i = \lambda_{i,0} + \sum_k \epsilon^k \lambda_{i,k}$$

$$\vec{v}_i = \vec{v}_{i,0} + \sum_k \epsilon^k \vec{v}_{i,k} \qquad \|v_i\| = \|v_{i,k}\| = 1$$

$\Rightarrow$ 1st order variation is 0
$\Rightarrow \vec{v}_{i,1} \perp \vec{v}_{i,0}$

$$\lambda_i = \lambda_{i,0} + \epsilon (H_1)_{ii} + \epsilon^2 \sum_{j \neq i} \frac{|(H_1)_{ij}|^2}{\lambda_{i,0} - \lambda_{j,0}} + O(\epsilon^3)$$

$$\vec{v}_i = \vec{v}_{i,0} + \epsilon \sum_{j \neq i} \frac{(H_1)_{ij}}{\lambda_{i,0} - \lambda_{j,0}} \vec{v}_{j,0} + O(\epsilon^2)$$

Using this, take $M_0$ initial $X_1$ Wigner

$$M = M_0 + \sqrt{dt}\, X_1$$

Using rotational inv of Wigner, WLOG $X_0$ diag

$$(X_1)_{ii} \sim N(0, \tfrac{2}{N})$$

$$(X_1)_{ji} \sim N(0, 1/N)$$

fluctuations negligible over $dt \to 0 \Rightarrow$ deterministic

$$\Rightarrow |X_{ij}|^2 = \frac{1}{N} + O(\tfrac{1}{N})$$

$$d\lambda_i = \underbrace{\sqrt{\frac{2}{\beta N}}\, dB_i}_{O(\lambda)} + \frac{1}{N} \underbrace{\sum_{j \neq i} \frac{dt}{\lambda_i - \lambda_j}}_{O(\lambda^2)}$$

$\leftarrow$ b.c. $|(X_1)_{ji}|^2$ can be treated deterministically

$\leftarrow$ coulomb force

$$d\vec{v}_i = \frac{1}{\sqrt{N}} \sum_{j \neq i} \frac{dB_{ij}}{\lambda_i - \lambda_j} v_j - \frac{1}{2N} \sum_{j \neq i} \frac{dt}{(\lambda_i - \lambda_j)^2} v_i$$

$(H_1)_{ij}$ $\quad |(H_1)_{ij}|^2$

Perturbation theory becomes exact. Works at any $N$

$dB_{ij} = dB_{ji}$ otherwise independent

**Ex 9.1.1**

$$d\lambda_i = \sqrt{\frac{2}{N}}\, dB_i + \frac{1}{N} \sum_{j \neq i} \frac{dt}{\lambda_i - \lambda_j}$$

$$F(\{\lambda_i\}) = \frac{1}{N}\sum_i \lambda_i^2$$

a) $$dF = \frac{\partial F}{\partial \lambda_i} d\lambda_i + \frac{2}{N}\frac{1}{2}\frac{\partial^2 F}{\partial \lambda_i} dt \cdot N$$

$$= \frac{1}{N}\sum_i 2\lambda_i\left[\sqrt{\frac{2}{N}}\, dB_i + \frac{1}{N}\sum_{j\neq i}\frac{dt}{\lambda_i - \lambda_j}\right] + \frac{2}{N}dt$$

$$= \frac{1}{N}\sum_i 2\lambda_i\sqrt{\frac{2}{N}}\, dB_i + \frac{1}{N^2}\sum_{i,j\neq i}\frac{2\lambda_i}{\lambda_i - \lambda_j} dt + \frac{2}{N}dt$$

$$= \frac{1}{N}\sum_i 2\lambda_i\sqrt{\frac{2}{N}}\, dB_i + \left(\frac{N(N-1)}{N^2} + \frac{2}{N}\right) dt$$

b) $$E[dF] = \frac{N+1}{N} dt$$

$$\Rightarrow F(t) = F(0) + \frac{N+1}{N} t$$

## 9.2 Itô Calculus

$$dX_{kk} = \sqrt{\frac{2}{N}}\, dB_{kk} \qquad dX_{k\ell} = \sqrt{\frac{1}{N}}\, dB_{k\ell}$$

Taking $X_0 = \mathrm{diag}(\lambda_1(0), \dots, \lambda_N(0))$

$$d\lambda_i = \sum_k \frac{\partial \lambda_i}{\partial X_{kk}}\sqrt{\frac{2}{N}}\, dB_{kk} + \sum_{k<\ell}\frac{\partial \lambda_i}{\partial X_{k\ell}}\sqrt{\frac{1}{N}}\, dB_{k\ell} + \sum_k \frac{\partial^2 \lambda_i}{\partial X_{kk}^2}\frac{dt}{N} + \sum_{k<\ell}\frac{\partial^2 \lambda_i}{\partial X_{k\ell}^2}\frac{dt}{2N}$$

$$X_0 + \delta X \rightsquigarrow \begin{pmatrix} \lambda_k & \delta X_{k\ell} \\ \delta X_{k\ell} & \lambda_\ell \end{pmatrix} \Rightarrow \frac{\lambda_k + \lambda_\ell}{2} \pm \frac{\lambda_k - \lambda_\ell}{2}\sqrt{1 + \frac{4(\delta X_{k\ell})^2}{(\lambda_k - \lambda_\ell)^2}}$$

$$\begin{aligned} \lambda_k &\to \lambda_k + \frac{(\delta X_{k\ell})^2}{\lambda_k - \lambda_\ell} \\ \lambda_\ell &\to \lambda_\ell - \frac{(\delta X_{k\ell})^2}{\lambda_k - \lambda_\ell} \end{aligned} \Rightarrow \frac{\partial \lambda}{\partial X_{k\ell}} = 0 \qquad \frac{\partial^2 \lambda_i}{\partial X_{k\ell}^2} = \frac{2\delta_{ik} - 2\delta_{i\ell}}{\lambda_k - \lambda_\ell}$$

$$\Rightarrow d\lambda_i = \sqrt{\frac{2}{N}}\, dB_i + \frac{1}{N}\sum_{j\neq i}\frac{dt}{\lambda_i - \lambda_j}$$ } as before!

*

## 9.3 Dyson BM for the Resolvent

$$M_t = M_0 + X_t$$

$$g_N(z, \{\lambda_i\}) = \frac{1}{N} \sum_i \frac{1}{z-\lambda_i}$$

$$\frac{\partial g_N}{\partial \lambda_i} = \frac{1}{N} \frac{1}{(z-\lambda_i)^2}, \quad \frac{\partial^2 g}{\partial \lambda_i^2} = \frac{2}{N} \frac{1}{(z-\lambda_i)^3}$$

$$dg_N = \frac{dg_N}{d\lambda_i} d\lambda_i^* + \frac{1}{2} \frac{\partial^2 g}{\partial \lambda_i^2} \frac{2}{N} dt$$

$$= \frac{1}{N} \sqrt{\frac{2}{N}} \sum_i \frac{dB_i}{(z-\lambda_i)^2} + \underbrace{\frac{1}{N^2} \sum_{i,j\neq i} \frac{dt}{(z-\lambda_i)^2 (\lambda_i - \lambda_j)}} + \frac{2}{N^2} \sum_i \frac{dt}{(z-\lambda_i)^3}$$

$$= \frac{1}{2N^2} \sum_{i,j\neq i} \left[ \frac{1}{(z-\lambda_i)^2 (\lambda_i - \lambda_j)} + \frac{1}{(z-\lambda_j)^2 (\lambda_j - \lambda_i)} \right]$$

$$= \frac{1}{2N^2} \sum_{i,j\neq i} \frac{2z - \lambda_i - \lambda_j}{(z-\lambda_i)^2 (z-\lambda_j)^2} = \frac{1}{N^2} \sum_{j\neq i} \frac{1}{(z-\lambda_i)(z-\lambda_j)^2}$$

$$= \frac{1}{N^2} \sum_{i,j} \frac{1}{(z-\lambda_i)(z-\lambda_j)^2} - \frac{1}{N^2} \sum_j \frac{1}{(z-\lambda_j)^3}$$

$$= -g_N \frac{\partial g_N}{\partial z} + \frac{1}{2N} \frac{\partial^2 g_N}{\partial z^2}$$

$$\Rightarrow dg_N = \frac{1}{N} \sqrt{\frac{2}{N}} \sum_i \frac{dB_i}{(z-\lambda_i)^2} - g_N \frac{\partial g_N}{\partial z} dt + \frac{1}{2N} \frac{\partial^2 g_N}{\partial z^2} dt$$

$$\Rightarrow \mathbb{E}\, dg_N = -\mathbb{E}\left[ g_N \frac{\partial g_N}{\partial z} \right] dt + \underbrace{\frac{1}{2N} \frac{\partial^2}{\partial z^2} \mathbb{E}\, g_N \, dt}_{\text{viscosity}} \quad \Big\} \text{ exact } \forall N$$

$$\Rightarrow \underbrace{\frac{\partial}{\partial t} g = -g g'}_{\text{inviscid Burgers'}} \quad \text{as } N \to \infty$$

→ can develop singularities
→ needs a viscosity term to regularize it (exactly the last term, dropped)

### 9.3.2 Evolution of Resolvent

$$G_t = (z\mathbb{1} - M_t)^{-1} \qquad M_t = M_0 + X_t$$

$$dG_{ij} = \sum_{kl} \underbrace{\frac{dG_{ij}}{dM_{kl}}} dX_{kl} + \frac{1}{2}\sum_{klmn} \frac{\partial^2 G_{ij}}{\partial M_{kl}\,\partial M_{mn}} \underbrace{\mathrm{Cov}(X_{kl}, X_{mn})} dt$$

b/c $M$ symmetric: $\frac{1}{2}[G_{ik}G_{jl} + G_{jk}G_{il}]$

$\frac{1}{N}(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm})$

$$\Rightarrow \frac{\partial^2 G_{ij}}{\partial M_{kl}\,\partial M_{mn}} = \frac{1}{4}\left[(G_{im}G_{kn} + G_{in}G_{km})G_{jl} + \cdots\right]$$

$$\Rightarrow dG_{ij} = \sum_{kl} G_{ik}G_{jl}\, dX_{kl} + \frac{1}{N}\sum_{kl}(G_{ik}G_{kl}G_{jl} + G_{il}G_{jl}G_{kk})\,dt$$

$$\Rightarrow \mathbb{E}\left[\frac{dG}{dt}\right] = \underbrace{\frac{1}{N}\mathrm{Tr}\, G}_{g}\ \underbrace{\mathbb{E}G^2}_{-\partial_z \mathbb{E}G} + \underbrace{\frac{1}{N}\mathbb{E}G^3}_{\frac{1}{2}\frac{1}{N}\partial_z^2 \mathbb{E}G}$$

$$\Rightarrow \mathbb{E}\frac{dG}{dt} = -g\,\partial_z \mathbb{E}G + \underbrace{\frac{1}{2N}}_{\to 0 \text{ as } N\to\infty} \partial^2 \mathbb{E}G$$

linear in $G$ if $g$ is known

## 9.4 DBM with $V$

$$P(\{\lambda_i\}) = Z^{-1}\exp\left[-\frac{\beta}{2}\left(\sum_i NV(\lambda_i) - \sum_{j\neq i}\log|\lambda_i - \lambda_j|\right)\right]$$

be wary of the factor of 2

Langevin $\sigma dB + \frac{\sigma^2}{2}\frac{d}{dx}\log P(x)dt$

$$\Rightarrow d\lambda_k = \sqrt{\frac{2}{N}}\, dB_k + \frac{1}{N}\left(-\frac{\beta}{2}NV'(\lambda_k) + \sum_{j\neq k}\frac{\beta}{\lambda_k - \lambda_j}\right)dt$$

$\sigma^2 = \frac{2}{N}$

$$= \sqrt{\frac{2}{N}}\, dB_k + \left[\frac{1}{N}\sum_{j\neq k}\frac{1}{\lambda_k - \lambda_j} - \frac{\beta}{2}V'(\lambda_k)\right]dt$$

Modified Burgers':

$V(\lambda) = \frac{\lambda^2}{2}$, $\beta = 1$ $\Rightarrow$ $\frac{\partial g}{\partial t} = -g\,\partial_z g + \frac{1}{2}\partial_z(zg)$

Ex 9.4.1

$$F_k = \frac{1}{N}\sum_i \lambda_i^k$$

a) $$dF_k = \frac{1}{N}\sum_i k\lambda_i^{k-1} d\lambda_i + \frac{1}{2}\sum_i k(k-1)\lambda^{k-2} dt$$

$$= \frac{1}{N}\sum_i k\lambda_i^{k-1}\left[\sqrt{\tfrac{2}{N}}\,dB_i + \left(\frac{1}{N}\sum_{j\neq i}\frac{\beta}{\lambda_i-\lambda_j} - \frac{\beta}{2}V'(\lambda_k)\right)dt\right] + \frac{2}{N}\,\frac{1}{2N}\sum_i k(k-1)\lambda^{k-2}\,dt$$

b) $$\mathbb{E}\frac{dF_2}{dt} = \frac{1}{N^2}\underbrace{\sum_{i\neq j}\frac{2\lambda_i}{\lambda_i-\lambda_j}}_{N(N-1)} - \frac{1}{N}\sum_i \lambda_i V'(\lambda_i) + \frac{2}{N}\underbrace{\sum_i\frac{1}{N}}_{1} dt$$

$\beta = 1$

$$= 1 + \mathbb{E}\frac{1}{N}\sum_i \lambda_i V'(\lambda_i) + 1/N$$

c) For Wigner $V'(x) = x$

$$\Rightarrow 0 = 1 + \frac{1}{N} - \mathbb{E}F_2$$

$$\Rightarrow \mathbb{E}F_2 = 1 + 1/N$$

d) For general $V$ we have as $N\to\infty$

$$1 = \mathbb{E}\left[\frac{1}{N}\sum_i \lambda_i V'(\lambda_i)\right] = \tau[XV'(X)]$$

e) For Wishart:

$$V(x) = q^{-1}x + (1-q^{-1})\log x$$

$$\Rightarrow V'(x) = q^{-1} + \frac{1-q^{-1}}{x}$$

$$\Rightarrow xV'(x) = q^{-1}x + (1-q^{-1})$$

$$1 = \tau(q^{-1}x) + 1 - q^{-1}$$

$$\Rightarrow q^{-1}\tau(X) = q^{-1} \quad\Rightarrow\quad \tau(X) = 1$$

f) $$0 = \mathbb{E}\frac{dF_{k+1}}{dt} = \frac{1}{N}\sum_i k\lambda_i^k\left(\frac{1}{N}\sum_{j\neq i}\frac{\beta}{\lambda_i-\lambda_j} - \frac{\beta}{2}V'(\lambda_i)\right) + \cancel{\frac{2}{N}\frac{1}{2N}\sum_i (k+1)k\,\lambda^{k-1}} \to 0 \; dt$$

$$\Rightarrow \tau(V'(X)X^k) = \frac{1}{N}\sum_i V'(\lambda_i)\lambda_i^k = \frac{2}{N^2}\sum_{j\neq i}\frac{\lambda_i^k}{\lambda_i-\lambda_j}$$

$$= \frac{1}{N^2}\sum_{\ell=0}^{k-1}\sum_{i\neq j} \lambda_i^{\ell}\lambda_j^{k-\ell-1}$$

$k=2 \Rightarrow \tau(V'(X)X^2) = \frac{2}{N^2}\sum_{i\neq j}\lambda_j = \frac{2(N-1)}{N^2}\sum \lambda_i$

$\to 2\tau(X)$

$k=3 \Rightarrow \tau(V'(X)X^3) = \frac{2}{N^2}\sum_{i\neq j}\lambda_i^2 + \frac{1}{N^2}\sum_{i\neq j}\lambda_i\lambda_j$

$= 2\tau[X^2] + \tau[X]^2$

g) For $V' = x \Rightarrow \tau(X^{k+1}) = \sum_{\ell=0}^{k-1}\tau(X^\ell)\,\tau(X^{k-\ell-1})$ } Catalan

## 9.4.2 Fokker Planck for DBM:

$$\partial_t P = -\nabla\cdot(\vec{F}P) + \frac{\sigma^2}{2}\nabla^2 P$$

$\sigma^2 = \frac{2}{N} \Rightarrow \dot{P} = \frac{1}{N}\sum_j \frac{\partial}{\partial\lambda_j}\left[\frac{\partial P}{\partial \lambda_j} - F_j P\right]$

$\frac{\partial}{\partial\lambda_k}\sum_{ij}\log|\lambda_i - \lambda_j|$

$= \sum_{ij}\frac{\delta_{ik}-\delta_{jk}}{\lambda_i-\lambda_j}$

$= \sum_j \frac{1}{\lambda_k-\lambda_j} - \sum_i \frac{1}{\lambda_i-\lambda_k}$

$= 2\sum_j \frac{1}{\lambda_k-\lambda_j}$

joint force now (no longer cumulative)

$F = -\nabla\tilde{V}/2$

$\tilde{V} = \frac{\beta}{N\sigma^2}\left[-\sum_{\substack{j\\ j\neq i}}\log|\lambda_i-\lambda_j| + N\sum_i V(\lambda_i)\right]$

$\Rightarrow F_i = \beta\left[\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j} - \frac{N\lambda_i}{2}\right]$

$$P(\{\lambda_i\};t) = \exp\Big[\underbrace{\frac{\beta}{4}\Big[\sum_{\substack{i,j\\ j\neq i}}\log|\lambda_i-\lambda_j| - N\sum_i \frac{\lambda_i^2}{2}\Big]}\Big]\, W(\{\lambda_i\};t)$$

$u_i = \int F/2 \qquad u' = F/2$

$e^u\dot{W} = \frac{1}{N}\sum_j \frac{\partial}{\partial\lambda_j}\left[\frac{\partial}{\partial\lambda_j}(e^u W) - F_j e^u W\right]$

$= \frac{1}{N}\sum_j e^u W_j'' + \cancel{2u' e^u W'} + u_j'' e^u W - \cancel{F_j e^u W'} - F_j' e^u W - F_j u' e^u W$

$$\Rightarrow \dot{W} = \frac{1}{N}\sum_i W'' - \underbrace{(-U_i'' + F_i' + F_i U_i')}_{V_i} W$$

$$V_i = \frac{1}{2}F_i' + \frac{F^2}{2}$$

$$V_i = -\frac{\beta}{2}\left[\sum_{j\neq i}\frac{1}{(\lambda_i-\lambda_j)^2} + \frac{N}{2}\right] + \frac{\beta^2}{2}\left[\frac{N\lambda_i}{2} - \sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}\right]^2$$

1) $$\frac{1}{N}\times 2\times\left(-\frac{\beta^2}{4}\frac{N}{2}\right)\times\sum_{\substack{i,j\\ j\neq i}}\frac{\lambda_i}{\lambda_i-\lambda_j} = -\frac{\beta^2}{4}\,\frac{1}{2}\,\frac{N(N-1)}{2}\times 2 = -\frac{\beta^2 N(N-1)}{8}$$

cancel $i$ sum

2) $$\sum_{\substack{j\neq i\\ k\neq i}}\frac{1}{\lambda_i-\lambda_j}\,\frac{1}{\lambda_i-\lambda_k} = \cancelto{0}{\sum_{j\neq k\neq i}\frac{1}{\lambda_i-\lambda_j}\,\frac{1}{\lambda_i-\lambda_k}} + \sum_{\substack{i,j\\ j\neq i}}\frac{1}{(\lambda_i-\lambda_j)^2} \Rightarrow \frac{\beta^2}{4}\sum_{j\neq i}\frac{1}{(\lambda_i-\lambda_j)^2}$$

3) $$\frac{\beta^2}{4}\,\frac{N^2\lambda_i^2}{4} = \frac{\beta^2N^2\lambda_i^2}{16}$$

$$\Rightarrow V_i = \frac{\beta^2N^2\lambda_i^2}{16} - \frac{\beta(2-\beta)}{4}\underbrace{\sum_{j\neq i}\frac{1}{(\lambda_i-\lambda_j)^2}}_{\text{Potential}} - \frac{N\beta}{4}\left(1+\frac{\beta(N-1)}{4}\right)$$

$$\Rightarrow \quad \frac{\partial W}{\partial t} = -\Gamma W_\Gamma$$

$$\frac{2}{N}\sum_{j=1}^{N}\left(-\frac{1}{2}\frac{\partial^2}{\partial\lambda_j^2} + \frac{1}{2}V_j\right)W_\Gamma = \Gamma W_\Gamma$$ ← eval eq'n

"Real Schrödinger Equation"

Calogero Model

$$\Gamma(n_1\cdots n_N) = \frac{\beta}{2}\left(\sum_i n_i - \frac{N(N-1)}{2}\right) \qquad 0\le n_1<\cdots<n_N$$

$W_\Gamma = 0$ when $\lambda_i = \lambda_j$ $\Rightarrow$ $W$ is fermionic

$n_1 = 0 \; \cdots \; n_N = N-1 \Rightarrow \Gamma = 0$

next excitation is

$n_1 = 0 \cdots n_{N-1} = N-2 \quad n_N = N \Rightarrow \Gamma = \frac{\beta}{2}$

$\tau_{eq} = 2/\beta \sim O(1)$

$\Rightarrow$ converges in $O(1)$ time!

$\beta = 2 \Rightarrow$ ;

## 9.5 Karlin McGregor

Brownian motion:

$$p(y,t|x) = \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{(x-y)^2}{2t}\right]$$

$$\frac{\partial p}{\partial t} = -\frac{1}{2}\frac{1}{\sqrt{2\pi t}}\left[\frac{1}{t} - \frac{(x-y)^2}{t^2}\right]\exp[\cdots]$$

$$\frac{\partial^2 p}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{1}{\sqrt{2\pi t}}\left(\frac{x-y}{t}\right)\exp\cdots\right] = \frac{1}{\sqrt{2\pi t}}\left[\left(\frac{x-y}{t}\right)^2 - \frac{1}{t}\right]$$

$$\Rightarrow \frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial y^2} \checkmark$$

$N$ indep Brownian motions starting at $\vec{x} = (x_1 \cdots x_N)$

$x_1 \geq \cdots \geq x_N$

$$P_{KM}(\vec{y},t|\vec{x}) = \left|\det p(y_i,t|x_j)\right|$$

sum of terms, each is a product $\Pi$ involving only one $y_i$ for each $i$

$$\Rightarrow \frac{\partial}{\partial t}\Pi = \frac{1}{2}\sum_i \frac{\partial^2 \Pi}{\partial y_i^2}$$

$$\Rightarrow \frac{\partial}{\partial t}P_{KM} = \frac{1}{2}\sum_i \frac{\partial P_{KM}}{\partial y_i^2}$$

Survival probability $\quad p(t|\vec{x}) := \int d\vec{y}\, p(\vec{y};t|\vec{x})$

decreases w/ time

can show $p(t|\vec{x}) \propto \sqrt{t}^{\,-N(N-1)/2}$

$P(\vec{y},t|\vec{x}) =: \mathbb{P}(\text{end at } \vec{y} \text{ @ } t \mid \text{start @ } \vec{x} \text{ \& never intersect})$

Not proven in book $\Rightarrow$ $= \dfrac{\Delta(x)}{\Delta(y)} P_{KM}(\vec{y},t|\vec{x})$

$\Delta(x) := \prod_{i<j} (x_i - x_j)$

Without confining potential @ $\beta=2$ this is

$\exp\left[\frac{1}{2}\sum_{i,j} \log|\lambda_i - \lambda_j|\right] W = \Delta(\vec{x})\, W(\{\lambda_i\},t)$

$W$ obeys the diffusion equation here
& vanishes linearly when two eigs meet

$\Rightarrow$ same as $P_{KM}$ w/ $\vec{y} = \vec{\lambda}$

# Chapter 10: Addition of large random matrices

$A+B \Rightarrow$ what is $\rho_{A+B}(\lambda)$ in terms of $\rho_A, \rho_B$

When $A$ is Wigner, can use DBM from before

## 10.1 Wigner:

Burgers': $\partial_t g = -g\, \partial_z g$

$g_0 := g_{M_0}(z)$

Method of Characteristics:

$$g_t(z) = g_0(z - t g_t(z))$$

$$\partial_t g_t = g_0'(z - t g_t(z))\left[-g_t(z) - t\, \partial_t g_t(z)\right]$$

$$\Rightarrow \partial_t g_t = -\frac{g_t\, g_0'(z - t g_t(z))}{1 + t g_0'(z - t g_t(z))}$$

$$\partial_z g_t = g_0'(z - t g_t(z))(1 - t g_t'(z))$$

$$\Rightarrow \partial_z g_t = \frac{g_0'(z - t g_t(z))}{1 + t g_0'(z - t g_t(z))} = -g_t\, \partial_t g_t$$

Comes from Method of Characteristics

Eg $M_0 = 0 \Rightarrow g_0 = z^{-1}$

$$\Rightarrow g_t = [z - t g_t]^{-1}$$

def $\mathfrak{z}_t(g_t(z)) = z \rightarrow \mathfrak{z}_t$ inv of $g_t$

$$\Rightarrow g = g_t(z) = g_0(z - t g)$$

$$z = \mathfrak{z}_t(g)$$

$\Rightarrow \mathcal{Z}_0(g) = z - tg = \mathcal{Z}_+(g) - tg$

$\Rightarrow \mathcal{Z}_+(g) = \mathcal{Z}_0(g) + tg$ $\Leftarrow$ additive shift

Eg 2 $M_0$ Wigner w/ var $\sigma^2$ $\frac{1}{g(z)} = z - \sigma^2 g$

$$\mathcal{Z}_0(g) = \sigma^2 g + \frac{1}{g}$$

$$\Rightarrow \mathcal{Z}_+(g) = \mathcal{Z}_0(g) + tg = (\sigma^2 + t)g + \frac{1}{g}$$

Generally $B = M_+$ $A = M_0$

$$\mathcal{Z}_B(g) = \mathcal{Z}_A(g) + tg$$
$$= \mathcal{Z}_A(g) + \mathcal{Z}_{X_+}(g) - \frac{1}{g}$$

def $R(g) := \mathcal{Z}(g) - \frac{1}{g}$

$$\Rightarrow R_B(g) = R_A(g) + R_{X_+}(g)$$

Eg White Wishart

$$qzg^2 - (z - 1 + q)g + 1 = 0$$

$$\Rightarrow z(qg-1)g = (q-1)g - 1$$

$$\Rightarrow z = \frac{q-1}{qg-1} - \frac{1}{(qg-1)g}$$
$$= \frac{(q-1)g-1}{(qg-1)g} = \frac{1}{g} + \frac{1}{1-qg}$$

$$\Rightarrow \mathcal{Z}(g) = \frac{1}{g} + \frac{1}{1-qg}$$

$$\Rightarrow R_W = \frac{1}{1-qg}$$

Ex 10.1.1

$$g(z) = \tau((z\mathbb{1}-M)^{-1}) = \int_{\operatorname{supp}\rho} \frac{\rho(\lambda)d\lambda}{z-\lambda}$$

$$g(z) = \sum_{n=0}^{\infty} \frac{m_n}{z^{n+1}} \qquad m_0 = 1$$

a) $$g = \frac{1}{z} + \frac{m_2}{z^3} + \frac{m_3}{z^4} + \frac{m_4}{z^5} + \cdots$$

$$\Rightarrow z^4 g = z^3 + z^2 m_2 + z m_3 + m_4$$

$\epsilon = \frac{1}{z}$ $\Rightarrow$ $$g = \frac{1}{z} + \frac{m_1}{z^2} + \cdots$$

$$\Rightarrow \frac{1}{g} = \frac{z}{1+m_1/z}$$

$$\Rightarrow z = \frac{1}{g} + \frac{1}{g}\frac{m_1}{z}$$

$z \to \infty \Rightarrow g \to 0$ near $g \sim 0$ $\quad \mathfrak{Z}(g) \sim 1/g$

$$\mathfrak{Z}(g) - 1/g \sim \frac{m_1}{zg} \sim \frac{m_1}{1+m_1/z} \to 1 \Rightarrow R(0) = m_1$$

b) Line by line in pert theory $\qquad x_1 = 0$, $x_2 = m_2$

$$\Rightarrow z^{(2)} = \frac{1}{g} + x_2 g$$

$$g^{(2)} = \frac{1}{z} + \frac{m_2}{z^3}$$

$$\Rightarrow z = \frac{1}{1/z + m_2/z^3} + \frac{x_2}{z} + \frac{x_2}{z^3} = z\left(1-\frac{m_2}{z^2}\right) + \frac{x_2}{z} + \cancel{\frac{x_2}{z^3}}$$

$$\Rightarrow m_2 = x_2$$

determined by: $\frac{1}{z}$, $\frac{1}{z^2}$, $\frac{1}{z^3}$

$$z = \frac{1}{g} + m_2 g + x_3 g^2 + x_4 g^3$$

$$g = \frac{1}{z} + \frac{m_2}{z^3} + \frac{m_3}{z^4}$$

$$\Rightarrow z = \frac{z}{1+\frac{m_2}{z^2}+\frac{m_3}{z^3}+\frac{m_4}{z^4}} + m_2\left(\frac{1}{z}+\frac{m_2}{z^3}+\frac{m_3}{z^4}\right) + x_3\left(\frac{1}{z}+\frac{m_2}{z^3}+\frac{m_3}{z^4}\right)^2 + x_4\left(\frac{1}{z}\right)^3$$

$$= z - \cancel{\frac{m_2}{z}} - \frac{m_3}{z^2} - \frac{m_4}{z^3} + \frac{m_2^2}{z^3} + \cancel{\frac{m_2}{z}} + \frac{m_2^2}{z^3} + \frac{m_3 m_2}{z^4} + \frac{x_3}{z^2} + \cdots + \frac{x_4}{z^3}$$

$\kappa_3 = m_3$

$\kappa_4 = m_4 - 2m_2^2$ ⇐ Free cumulant!

## Ex 10.1.2

$$g_{\alpha A}(z) = \tau\left((z\mathbb{1} - \alpha A)^{-1}\right) = \alpha^{-1} g_A(\alpha^{-1} z)$$

$$g_{A+b\mathbb{1}}(z) = g_A(z-b)$$

$$\mathfrak{z}_A(g_A) = z$$

$$\mathfrak{z}_A(g_A(z-b)) = z-b \qquad\qquad \mathfrak{z}_A(g_A(\alpha^{-1}z)) = \alpha^{-1} z$$

$$\mathfrak{z}_A(g_{A+b\mathbb{1}}(z)) = z-b \qquad\qquad \alpha\,\mathfrak{z}_A(\alpha\, g_{\alpha A}(z)) = z$$

$$\Rightarrow \mathfrak{z}_{A+b\mathbb{1}}(g_{A+b\mathbb{1}}(z)) = \mathfrak{z}_A(g_{\dots}) + b \qquad\qquad \mathfrak{z}_{\alpha A}(g_{\alpha A}) = z$$

$$\Rightarrow \boxed{\mathfrak{z}_{A+b\mathbb{1}}(g) = \mathfrak{z}_A(g) + b} \qquad\qquad \boxed{\mathfrak{z}_{\alpha A}(g) = \alpha\ \ {}_A(\alpha g)}$$

$$\Rightarrow R_{A+b\mathbb{1}} = R_A + b \qquad\qquad \Rightarrow R_{\alpha A} = \mathfrak{z}_{\alpha A} - \frac{1}{g}$$

$$= \alpha\,\mathfrak{z}_A(\alpha g) - \frac{\alpha}{\alpha g} = \alpha R_A(\alpha g)$$

## Ex 10.1.3

$M$ orthogonal symmetric
$X$ Wigner

$E = M + X$

a) evals of $M$ are $\pm 1$ each with $P = 1/2$

$$\Rightarrow \rho(\lambda) = \frac{1}{2}\delta(\lambda \pm 1)$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{z-1} + \frac{1}{z+1}\right) = \frac{z}{z^2-1} = g_M(z)$$

b) $$g_+(z) = g_0(z - t g_+(z))$$

$$= \frac{z - t g_+(z)}{(\dots)^2 - 1}$$

$(z - tg_t(z))$

$\Rightarrow g_t(z)(z - tg_t(z))^2 - g_t(z) = z - tg_t(z)$

$t^2 g^3 - 2ztg^2 + (z^2-1)g = z - tg$

c) $g_0 = \frac{z}{z^2-1} \Rightarrow z^2 g_0 - z - g_0 = 0$

$\Rightarrow z = \frac{1}{2g_0} \pm \frac{\sqrt{1+4g_0^2}}{2g_0}$

$\Rightarrow z_0(g) = \frac{1 + \sqrt{1+4g^2}}{2g}$ $\Leftarrow$ + bc $\frac{1}{g}$ as $g \to \infty$

d) $z_t(g) = z_0(g) + tg$

$\Rightarrow z = \frac{1}{2g} + \frac{\sqrt{1-4g^2}}{2g} + tg$

$(2zg - 1 - 2tg^2)^2 = 1 - 4g^2$

$4z^2g^2 + \cancel{1} + 4t^2g^4 - 4zg + 4tg^2 - 8tzg^3 = \cancel{1} - 4g^2$

$\Rightarrow z^2 g + t^2 g^3 - z + tg - 2tzg^2 = -g$

$t^2 g^3 - 2tzg^2 + (z^2-1)g = z - tg$ $\Leftarrow$ almost identical

e) Need nonreal $g$ for real $z$

$z = 0 \Rightarrow t^2 g^2 - 1 + t$

$g = \pm\sqrt{\frac{1-t}{t}}$

$\Rightarrow t > 1 \Rightarrow$ non-real $\Rightarrow$ eig density @ 0

$\Rightarrow \sigma^2 > 1$

f) $\sigma^2 = 1$ $(t=1)$ $\Rightarrow g^3 - 2g^2 z + gz^2 - z$

$\Rightarrow \Delta = -27z^2 + 4z^4 = z^2(4z^2 - 27)$

$(z=0)\times 2$ $\quad z = \pm\frac{\sqrt{27}}{2}$

$\Delta < 0 \Rightarrow |z| < \frac{\sqrt{27}}{2} = \frac{3\sqrt{3}}{2}$

```
In[515]:= -g + g t + g^3 t^2 - z - 2 g^2 t z + g z^2 /. t → 1
Out[515]= g^3 - z - 2 g^2 z + g z^2

In[520]:= a = 1; b = -2 z; c = z^2; d = -z;
          18 a b c d - 4 b^3 d + b^2 c^2 - 4 a c^3 - 27 a^2 d^2
Out[521]= -27 z^2 + 4 z^4
```

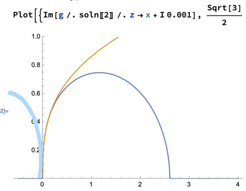
g) $\sigma^2 = 1$ has $g(0) \sim 0$

$\Rightarrow g = z^{1/3} + O(z)$

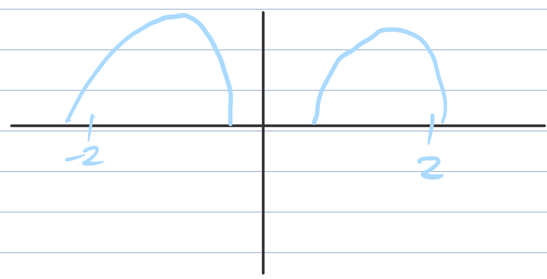
$\Rightarrow g(x) = \dfrac{\frac{\sqrt{3}}{2}\sqrt[3]{|x|}}{\pi}$ ?? $\leftarrow$ $\operatorname{Im} \sqrt[3]{-\epsilon} \sim \frac{\sqrt{3}}{2}\sqrt[3]{|\epsilon|}$

h) $t = 1$:

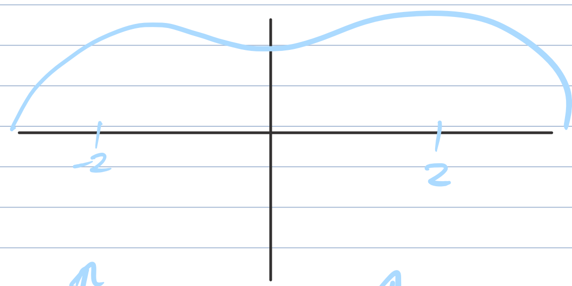
```
In[561]:= soln = Solve[(-g + g t + g^3 t^2 - z - 2 g^2 t z + g z^2 /. t → 1) == 0, g];
          Plot[{Im[g /. soln[[2]] /. z → x + I 0.001], Sqrt[3]/2 x^(1/3)}, {x, -0.5, 4}, PlotRange → {0, 1}]
Out[562]=
```



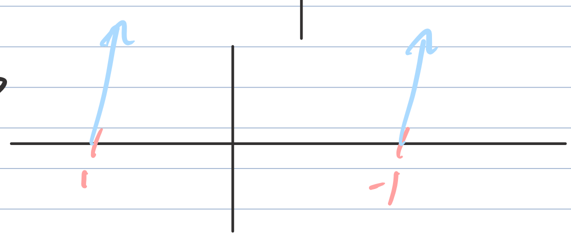

$t = 1/2 \Rightarrow$




$t = 2 \Rightarrow$




$t = 0 \Rightarrow$



## 10.2 Generalization to Non-Wigner

If $A, B$ have common eigs

$A+B$ has $\nu_i = \lambda_i + \mu_i$

Assume evecs are Haar distributed ⇒ unlikely overlap

⇒ define "free addition"

$$C = B + OAO^T, \quad O \sim \text{Haar}(O(N))$$

Candidate for CGF:

$$I(X,T) := \left\langle \exp\left(\frac{N}{2} \operatorname{Tr} TOXO^T\right)\right\rangle_O$$

← depends only on evals of $X$

$$I(B + O_1 A O_1^T, T) = \left\langle \exp\left(\frac{N}{2} \operatorname{Tr} TOBO^T + \frac{N}{2} \operatorname{Tr} TOO_1AO_1^TO^T\right)\right\rangle_O$$

$$= I(B,T)\, I(A,T)$$

⇒ $\log I$ is additive!

Start with $T = t\, vv^T$ rank 1

$$I(T,B) \simeq \exp\left[\frac{N}{2} H_B(t)\right]$$

$$H_B := \lim_{N\to\infty} \frac{2}{N} \log \left\langle \exp\left[\frac{tN}{2} \operatorname{Tr} vv^T OBO^T\right]\right\rangle_O$$

For $A,B$ randomly rotated: $H_C = H_A + H_B$

Calculate $H_B$: WLOG $B$ diag $\qquad OTO = \psi\psi^T$ ← random projector

$$Z_t(B) = \int \frac{d^N\psi}{(2\pi)^{N/2}}\, \delta(\|\psi\|^2 - Nt) \exp\left(\frac{1}{2}\psi^T B \psi\right)$$

$Z_t(0) \neq 1$ atm

$$\delta(x) = \int_{-\infty}^{\infty} e^{izx} \frac{dz}{2\pi} = \int_{-i\infty}^{i\infty} e^{-zx/2} \frac{dz}{4\pi i}$$

$$\Rightarrow Z_t(B) = \int_{\Lambda - i\infty}^{\Lambda + i\infty} \frac{dz}{4\pi i} \int \frac{d^N\psi}{(2\pi)^{N/2}} \exp\left[\frac{1}{2}\psi^T(B - z\mathbb{1})\psi + \frac{zNt}{2}\right]$$

$$= \int_{\Lambda - i\infty}^{\Lambda + i\infty} \frac{dz}{4\pi i} \exp\left[\frac{N}{2}\underbrace{\left(zt - \frac{1}{N}\sum_i \log(z-\lambda_i)\right)}_{F_t(z,B)}\right]$$

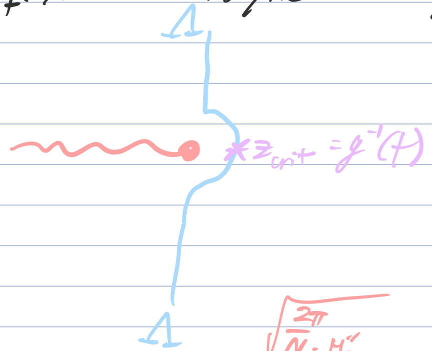
$$\partial_z F \Rightarrow t - \frac{1}{N}\sum_i \frac{1}{z-\lambda_i} \Rightarrow t - g_N^B(z) = 0$$

$$\Rightarrow z = g^{-1}(t)$$

For $x > \lambda_{max}$ $g$ is monotonically decreasing $\Rightarrow$ $t < \lambda_{max}$ works

likewise for $x < \lambda_{min}$ <u>but</u>

$F_t(z,B)$ is analytic in $z$ only for $Re(z) > \lambda_{max}$




$$\sqrt{\frac{2\pi}{N \cdot \frac{H''}{2}}}$$

$$\Rightarrow Z_t(B) = \frac{1}{4\pi}\sqrt{\frac{4\pi}{N\partial_z^2 F_t(\zeta(t),B)}} \exp\left[\frac{N}{2}\left[\zeta(t)t - \frac{1}{N}\sum_i \log \zeta(t) - \lambda_i(B)\right]\right]$$

$$= \frac{1}{2\sqrt{\pi N |g_B'(\zeta(t))|}}$$

$$B = 0 \Rightarrow g = z^{-1} \Rightarrow \zeta(t) = t^{-1}$$

$$\Rightarrow Z_t(0) = \frac{1}{2t\sqrt{\pi N}} \exp\left[\frac{N}{2}\left[1 + \log t\right]\right] \Rightarrow H_B(t) = \mathcal{H}_B(\zeta(t), t)$$

not super easy to work with $\Rightarrow$ $\mathcal{H}_B(z(t),t) = zt - 1 - \log t - \frac{1}{N}\sum_i \log(z-\lambda_i)$

Note $\partial_z \mathcal{H}(\zeta(t),t) = 0 \Rightarrow \frac{d}{dt} H_B(t) = \frac{d}{dt}\mathcal{H}(z,t) = \frac{\partial}{\partial t}\mathcal{H}(\zeta(t),t)$

$$= z - \frac{1}{t} = R_B(t)$$

$$\Rightarrow H_B(t) = \int_0^t dt' \, R_B(t')$$

$\leftarrow$ ensures $H(0)=0$

$H$ is additive $\Rightarrow R$ is too!

$$R_C(t) = R_A(t) + R_B(t) \quad \text{for } A,B \text{ relatively free}$$

Whenever rank $T \ll N$, can show

$$\Rightarrow I(T,B) \simeq \exp\left[\frac{N}{2}\sum_i H_B(t_i)\right] = \exp\left[\frac{N}{2}\mathrm{Tr}\, H_B(T)\right]$$

Needed $t$ small $\Rightarrow z_{crit} = \mathfrak{z}(t)$ large

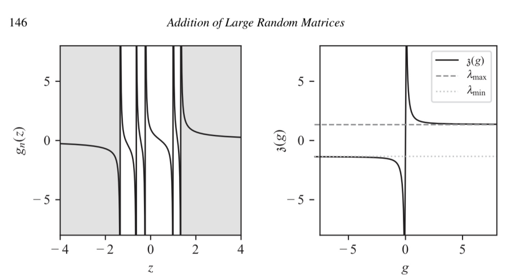
## 10.4 Invertibility of Stieltjes:

$$g_N^A(z) = \sum_i \frac{1}{z-\lambda_i}$$

For $z < \lambda_{min}$ or $z > \lambda_{max}$ $g_N$ is monotonically decreasing

$$g_N^A(z) = \frac{1}{z} + O\left(\frac{1}{z^2}\right)$$

$\Rightarrow g_N(z)$ is invertible for large $z$, behaves as $\frac{1}{g}$ + regular




$z > \lambda_{max} \Rightarrow g$ takes all values $>0$ exactly once
$z < \lambda_{min} \Rightarrow$ 〃 $<0$ 〃

$\Rightarrow \mathfrak{z}(g)$ exists $\forall g \neq 0$

$$\lim_{g\to-\infty} \mathfrak{z}(g) = \lambda_{min}$$
$$\lim_{g\to\infty} \mathfrak{z}(g) = \lambda_{max}$$

As $N\to\infty$ $g(z) = \frac{1}{z} + O\left(\frac{1}{z^2}\right)$ still
and $\mathfrak{z}(g) = 1/g$ + reg.

But now at eg $\lambda_+$ $g(z) = \int \frac{\rho(x)\,dx}{z-x}$

$\rho(\lambda) \sim (\lambda_+ - \lambda)^\theta \quad \theta > 0 \quad \Rightarrow \int \frac{(\lambda_+ - x)^\theta}{z - x} dx \Rightarrow \int (\lambda_+ - x)^{\theta - 1} dx$

Finite! at $z = \lambda_+$

Branch cut below $\lambda_+$

$\lambda_+$ is essential singularity for $g$ — $g(\lambda_+)$ is well-defined

(what do they really mean by essential here...)

$\Rightarrow \mathfrak{z}(g)$ exists for $g_- \leq g \leq g_+$

$\mathfrak{z}(g_\pm) = \lambda_\pm$

Wigner: $\lambda_\pm = \pm 2 \quad g_\pm = \pm 1 \Rightarrow \mathfrak{z}$ exists on $[-1, 1]$

## 10.4.3 Extending domain of $\mathfrak{z}(g)$

For e.g. HCIZ integral want to know $\mathfrak{z}$ beyond $g_\pm$

Wigner: $g + \frac{1}{g} - z = 0$

$\Rightarrow \mathfrak{z}(g) = g + 1/g \Leftarrow$ not the inverse of $g$ for $|g| > 1$

Wrong!

Realize we use $g$ as an approximator to $g_N$ for large $N$

For $z > \lambda_+$ this converges $g_N \to g$
but not on supp $\rho$

For finite $N$ there is $\lambda_{max} = \lambda_+$ at leading order in $1/N$

$\Rightarrow g_N(z) \approx g(z) + \frac{1}{N} \frac{1}{z - \lambda_{max}} \approx g(z) + \frac{1}{N} \frac{1}{z - \lambda_+}$

At any finite distance above $\lambda_+$, $g_N \to g$ as $N \to \infty$

$\lim_{z \to \lambda_+} g_N(z) \to \infty \Rightarrow \mathfrak{z}(g) = \lambda_+ \quad \forall g > g_+$

$$\Rightarrow \mathfrak{z}(g) = \begin{cases} \lambda_- & g < g_{min} \\ z_{bulk}(g) & g \in [g_{min}, g_{max}] \\ \lambda_+ & g \geq g_{max} \end{cases}$$



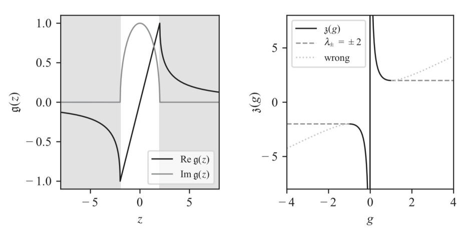



Figure 10.3 (left) The limiting function $\mathfrak{g}(z)$ for a Wigner matrix, a typical density that vanishes at its edges. The function is plotted against a real argument. In the white part of the graph, the function is ill defined and it is shown here for a small negative imaginary part of its argument. In the gray part ($z < \lambda_-$ and $z > \lambda_+$) the function is well defined, real and monotonic. It is therefore invertible. (right) The inverse function $\mathfrak{z}(g)$ only exists for $g_- \leq g \leq g_+$ and has a $1/g$ singularity at zero. The dashed lines show the extension of $\mathfrak{z}(g)$ to all values of $g$ that are natural when we think of $\mathfrak{g}(z)$ as the limit of $g_N(z)$ with maximal and minimal eigenvalues $\lambda_\pm$. The dotted lines indicate the wrong branch of the solution of $\mathfrak{z}(g) = g + 1/g$.

### 10.4.4 Large $t$ behavior of $I_t(B)$

$$I_t(B) := \left\langle \exp\left(\frac{N}{2} t\, \psi^T B \psi\right)\right\rangle_\psi$$

$$\exp\left[\frac{N}{2} t \lambda_{min}\right] < I_t(B) < \exp\left[\frac{Nt}{2} \lambda_{max}\right]$$

$$\Rightarrow H_B \leq t \lambda_{max}$$

For Wigner $R_W(t) = t \Rightarrow H_W = t^2/2$

but $\lambda_{max} = 2 \Rightarrow$ fails since $H_W > 2t$ for $t > 4$

But our calculation is valid only for $t < g_+ = g(2) = 1$

For $t > g_+$ $\zeta = \lambda_{max}$

$$\Rightarrow \frac{dH}{dt} = \begin{cases} 3(t) - 1/t = R_B(t) & t \leq g_{max} \\ \lambda_{max} - 1/t & t > g_{max} \end{cases}$$

Will later derive this w/ Replicas

## 10.5 Full-Rank HCIZ

$$I_\beta(A,B) := \int_{G(N)} dU \exp\left[\frac{\beta N}{2} \operatorname{Tr} AUBU^\dagger\right]$$

$G(N) = O(N), U(N), Sp(N)$
$\beta = 1, 2, 4$

For $\beta = 2$ the famous HCIZ result:

$$I_2(A,B) = \frac{c_N}{N^{N(N-1)/2}} \frac{\det(e^{N v_i \lambda_j})}{\Delta(A)\ \Delta(B)} \qquad c_N = \prod_{\ell}^{N-1} \ell!$$

Vandermonde dets

Can be obtained from Karlin-McGregor

### 10.5.1 Derivation

Interpret $e^{N \operatorname{Tr} AUBU^\dagger}$ for $\beta = 2$ as a diffusion propagator in the space of unitary matrices

$$P(B|A) \propto \underset{\text{det factor}}{N^{N^2/2}} \, \underset{\text{multidim gaussian}}{e^{-N/2 \, \mathrm{Tr}(B-A)^2}}$$

eigs follow $\quad dx_i = \sqrt{\frac{1}{N}}\, dB_i + \frac{1}{N}\sum_j \frac{dt}{x_i - x_j} \qquad \sigma^2 = 1/N \qquad \beta = 2$

$$P(\{\lambda_i\}|\{\nu_i\}) = \frac{\Delta(B)}{\Delta(A)} \underbrace{P(\lambda, t=1|\nu)}_{\det(p(\lambda_i, t=1|\nu_j))} \quad \star$$

$$\sqrt{\frac{N}{2\pi}}^{N^2} \begin{vmatrix} e^{-(\lambda_1-\nu_1)^2 N/2} & e^{-(\lambda_1-\nu_2)^2 N/2} & \cdots \\ e^{-(\lambda_2-\nu_1)^2 N/2} & \cdots & \\ e^{-(\lambda_N-\nu_1)^2 N/2} & \cdots & e^{-(\lambda_N-\nu_N)^2 N/2} \end{vmatrix} = \left(\frac{N}{2\pi}\right)^{N/2} e^{\frac{N}{2}(\sum \lambda_i^2 + \sum \nu_i^2)} \det\left(e^{N\lambda_i \nu_j}\right)$$

$$\overline{P(B|A)} := \frac{\int dU \; P(UBU^T|A)}{\Omega_N} = N^{N^2/2} e^{-N/2(\mathrm{Tr}\,A^2 + \mathrm{Tr}\,B^2)} \frac{I_2(A,B)}{\Omega_N}$$

$$\Omega_N = \int_{U(N)} dU$$

$B \to \lambda_i$

$dB \to d\lambda_i \quad \Delta^2(B)$

$$\Rightarrow P(\{\lambda_i\}|\{\nu_i\}) \propto N^{N^2/2} e^{-\frac{N}{2}(\mathrm{Tr}\,A^2 + \mathrm{Tr}\,B^2)} I_2(A,B)\Delta^2(B) \quad \star\star$$

$$\Rightarrow I_2(A,B) \propto N^{N(N-1)/2} \frac{\det(e^{N\lambda_i \nu_j})}{\Delta(A)\,\Delta(B)}$$

can recover $c_N$ from $A \to \mathbb{1}$

For $\nu_1 = t \quad \nu_{i \neq 1} = 0$ get rank 1 HCIZ:

$$I_2(t, B) = \frac{(N-1)!}{(Nt)^{N-1}} \sum_j \frac{e^{Nt\lambda_j}}{\prod_{k \neq j}(\lambda_j - \lambda_k)}$$

## 10.5.2 HCIZ at large N

DBM for $\overline{P(B|A)}$

Start at $\nu_i$ for $t=0$ end at $\lambda_i$

$$dx_i = \sqrt{\frac{1}{N}}\, dB_i - \partial_{x_i} V\, dt \qquad V(\{x_i\}) = -\frac{1}{N}\sum_{i<j} \log|x_i - x_j|$$

$$\Rightarrow P(\{x_i\}) = \frac{1}{Z}\exp\left[-\frac{N}{2}\int_0^1 dt \sum_i (\dot{x}_i + \partial_{x_i} V)^2\right] =: Z^{-1} e^{-N^2 S}$$

Neglecting Jac

Return to this

# Chapter 11: Free Probabilities

## 11.3.3 Additivity of the R-transform

Def: $g_A(z) = \sum_k \frac{\tau(A^k)}{z^{k+1}}$

$\Rightarrow R_A(g) := \mathfrak{z}_A(g) - 1/g$ $\leftarrow$ $\mathfrak{z}_A(g)$ formal power series satisfying $\mathfrak{z}_A(g_A(z)) = z$ to all orders

scalar $g$:

$$\tau(g\mathbb{1}) = g = g_A(\mathfrak{z}_A(g)) = \tau[(\mathfrak{z}_A(g) - A)^{-1}]$$

Def $gX_A = (z_A - A)^{-1} - g\mathbb{1}$

$$\Rightarrow A - z_A = -\frac{1}{g}(\mathbb{1} + X_A)^{-1}$$

Take $B$ free from $A$ $\quad z_B := \mathfrak{z}_B(g)$

$$B - z_B = -\frac{1}{g}(\mathbb{1} + X_B)^{-1}$$

$X_A, X_B$ also free $\Rightarrow A + B - z_A - z_B = -\frac{1}{g}(1+X_A)^{-1} - \frac{1}{g}(1+X_B)^{-1}$

$$= -\frac{1}{g}(1+X_A)^{-1}(2 + X_A + X_B)(1+X_B)^{-1}$$

$2 + X_A + X_B = (1+X_A)(1+X_B) + 1 - X_A X_B$

$$\Rightarrow A + B - z_A - z_B + \frac{1}{g} = -\frac{1}{g}(1+X_A)^{-1}(1 - X_A X_B)(1+X_B)^{-1}$$

$$\Rightarrow [A + B - (z_A + z_B - 1/g)]^{-1} = -g(1+X_B)(1 - X_A X_B)^{-1}(1+X_A)$$

$$= -g(1+X_B)\sum_k (X_A X_B)^k (1+X_A)$$

$$= \tau[A + B - (z_A + z_B - 1/g)]^{-1} = -g\,\tau(1) + \cancelto{0}{\tau(X_A X_B X_A X_B \cdots)}$$

$$g_{A+B}(z_A + z_B - g^{-1}) = g$$

$$\Rightarrow \mathfrak{z}_{A+B}(g) = \mathfrak{z}_A(g) + \mathfrak{z}_B(g) - 1/g$$

$$\Rightarrow R_{A+B}(g) = R_A(g) + R_B(g)$$

## 11.3.4 R-transform & Cumulants

$$R_A(g) = \sum_k x_k \, g^{k-1}$$

$$R_A(g) = z(g_A) - 1/g_A \Rightarrow z g_A(z) - 1 = g_A(z) R_A(g_A(z))$$

$$\sum_{k=1}^{\infty} \frac{m_k}{z^k} = \sum_{k=1}^{\infty} x_k \left[\frac{1}{z} + \sum_{\ell=1}^{\infty} \frac{m_\ell}{z^{\ell+1}}\right]^k$$

$\Rightarrow z^{-1}: \; m_1 = x_1 \quad \Rightarrow \quad m_1 = x_1$

$z^{-2}: \; m_2 = x_2 + x_1 m_1 \quad \Rightarrow \quad m_2 = x_2 + x_1^2$

$z^{-3}: \; m_3 = x_3 + 2x_2 m_1 + x_1 m_2 \quad \Rightarrow \quad m_3 = x_3 + 3x_2 x_1 + x_1^3$

$m_k = x_k +$ lower $x_\ell \cdot m_\ell$ combs

$\Rightarrow x_k(A) = \tau(A^k) +$ homogeneous products of lower order

+additive $\Rightarrow$ uniquely defines the cumulant

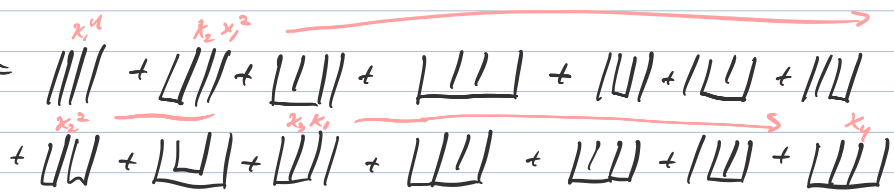
## 11.3.5 Cumulants from non-crossing

$$m_n = \sum_{\pi \in NC(n)} x_{\pi_{\ell_1}} \cdots x_{\pi_{\ell_n}}$$

$\pi$ is a non-crossing partition of $n$ elements

$$\sum_{k=1}^{\ell_\pi} \pi_k = n$$

Eg: $m_4 =$




$$= x_1^4 + 6x_2 x_1^2 + 2x_2^2 + 4x_3 x_1 + x_4$$

$$\Rightarrow m_n = \sum_{\ell=1}^{n} x_\ell \prod_{\substack{k_1 \cdots k_\ell = 0 \\ k_1 + \cdots + k_\ell = n-\ell}} m_{k_1} \cdots m_{k_\ell}$$

size of 1's partition

$$\Rightarrow \sum_{n=1}^{\infty} \frac{m_n}{z^n} = \sum_{n=1}^{\infty} \frac{1}{z^n} \sum_{l=1}^{n} \kappa_l \prod_{\cdots} m_{k_1} \cdots m_{k_l}$$

$$= \sum_{l=1}^{\infty} \kappa_l \left[ \sum_{k=0}^{\infty} \frac{m_k}{z^{k+1}} \right]^l = g(z)\, R(g(z))$$

For $z^{-n}$ need $\sum k_i + l = n$ w/ coeff $\kappa_l \; m_{\kappa_1} \cdots m_{\kappa_l}$

N.B at this stage there is nothing to do w/ freeness as we focus only on 1 variable A

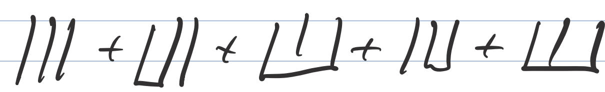
In commutative case

$$\sum_{n=0}^{\infty} \frac{m_n z^n}{n!} = \exp\left[ \sum_{r=1}^{\infty} \kappa_r \frac{(it)^r}{r!} \right]$$

## 11.3.6 Freeness as vanishing of mixed cumulants

$$\tau(A_1 \cdots A_n) = \sum_{\pi \in NC(n)} \underbrace{\kappa_\pi(A_1, \cdots, A_n)}_{\text{mixed cumulant}}$$

eg $\tau(A_1 A_2 A_3) =$



$$\kappa_1(A_1)\kappa(A_2)\kappa(A_3) + \kappa_2(A_1 A_2)\kappa_1(A_3) + \kappa_2(A_1 A_3)\kappa_1(A_1) + \kappa_1(A_1)\,\kappa_2(A_2 A_3) + \kappa_3(A_1, A_2, A_3)$$

A set of variables is free iff all of their mixed cumulants vanish

Free $\Rightarrow$ $\kappa$'s additive

$$\kappa_k(A+B, \cdots A+B) = \kappa_k(A) + \kappa_k(B)$$

## 11.3.7 CLT for Free Vars

Sum of K free identically distributed (FID) variables $\cdot\, 1/K \Rightarrow$ constant w/ some mean

Define Wigner var as $\kappa_2 > 0$, all other $\kappa_j = 0$

$$\Rightarrow R_W(g) = \kappa_2 g$$

$$\Rightarrow z = \kappa_2 g - 1/g$$

$$\Rightarrow g = \frac{z - z\sqrt{1 - \frac{4\kappa_2}{z^2}}}{2\kappa_2}$$ $\kappa_2 = \sigma^2 \Rightarrow$ Wigner w/ $\sigma^2$ (exists)

CLT says $\frac{1}{\sqrt{K}} \sum_{j=1}^{K} X_j$ $X_j$ are $\kappa_1 = 0$ $\kappa_2 > 0$ FID

$\rightarrow$ Wigner with $\kappa_2$

Again, pairwise-freeness is not enough

11.3.8

$$z_A(g) + R_B(g) = \underbrace{z_{A+B}(g)}_{:= z}$$

$$\Rightarrow z_A(g) = z_{A+B}(g) - R_B(g)$$

set $g = g_{A+B}(z) \Rightarrow z_A(g_{A+B}(z)) = z - R_B(g_{A+B}(z))$

$$\Rightarrow \boxed{g_{A+B}(z) = g_A(z - R_B(g_{A+B}(z)))}$$ "Subordination relation"

To get from $g_A(z)$ to $g_{A+B}(z)$, shift $z$ by $-R_B(g_{A+B}(z))$

## 11.4 Free Product

If $A, B$ are free & $\tau(A) = \tau(B) = 0$

$\tau((AB)^k) = \tau(ABAB\cdots) = 0 \Rightarrow$ trivial!

### 11.4.1 Low moments

$C = AB$ $A, B$ free $\tau(A) \neq 0$ $\tau(B) \neq 0$ WLOG $\tau(A) = \tau(B) = 1$

$$\tau(C) = \tau[\underbrace{(A - \tau(A))}_{\text{mean}}\underbrace{(B - \tau(B))}_{0 \text{ vars}}] + \tau(A)\tau(B) = 1$$

$\Rightarrow 0$

$$\tau(C) = \cancel{\kappa_2(A,B)}^{\,0} + \overset{1}{\kappa_1(A)}\, \overset{1}{\kappa_1(B)} = 1$$

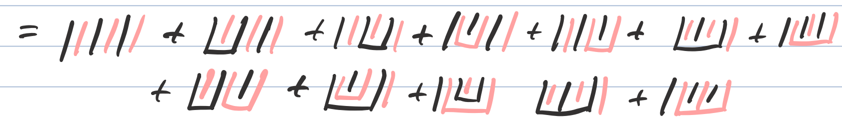
mixed cumulant

$$\tau(C^2) = \tau(ABAB) = \kappa_1(A)^2 \kappa_1(B)^2 + \kappa_2(A)\kappa_1(B)^2 + \kappa_2(B)\kappa_1(A)^2$$

$$= 1 + \kappa_1(A) + \kappa_2(B)$$

$$\Rightarrow \kappa_2(C) = \tau(C^2) - \tau(C)^2 = \kappa_2(A) + \kappa_2(B)$$

$$\tau(C^3) = \tau(ABABAB)$$



$$= 1 + 3\kappa_2(A) + 3\kappa_2(B) + 3\kappa_2(A)\kappa_2(B) + \kappa_3(A) + \kappa_3(B)$$

$$\Rightarrow \kappa_3(C) = \tau(C^3) - 3\tau(C^2)\tau(C) + 2\tau(C^3)$$

$$= \kappa_3(A) + \kappa_3(B) + 3\kappa_2(A)\kappa_2(B)$$

## 11.4.2 S-transform def'n

First T-transform:

$$t_A(\varsigma) = \tau[(1-\varsigma^{-1}A)^{-1}] - 1$$

$$= \varsigma\, g_A(\varsigma) - 1$$

$$= \sum_{k=1}^{\infty} \frac{m_k}{\varsigma^k}$$

same singularities as $g_A(z)$ except maybe at 0

$$\lim_{\eta \to 0^+} \operatorname{Im} t(x - i\eta) = \pi x \rho(x)$$

$g$ regular @ 0 $\Rightarrow t_A(0) = -1$

shifts from this indicate dirac mass there

$$t_A(\varsigma) = \tau\left[\frac{1}{1-\varsigma^{-1}A} - 1\right] = \tau\left[\frac{\varsigma^{-1}A}{1-\varsigma^{-1}A}\right] = \tau[A(\varsigma - A)^{-1}]$$

For $m_1 \neq 0$ $t_A$ is invertible for large $\zeta$

$\Rightarrow \zeta_A(t)$ s.t. $t_A(\zeta_A(t)) = t$

Define $S_A(t) := \dfrac{t+1}{t\,\zeta_A(t)}$

$S_{\mathbb{1}}(t)$ has $t_{\mathbb{1}}(\zeta) = \dfrac{1}{\zeta - 1} \Rightarrow \zeta_{\mathbb{1}}(t) = t^{-1}+1 \Rightarrow S_{\mathbb{1}}(t) = 1$

Identity is free wrt any var

$$t_{\alpha A}(\zeta) = \tau\left[\left(1-(\alpha^{-1}\zeta)^{-1}A\right)^{-1}\right] - 1 = t_A(\zeta/\alpha)$$

$\Rightarrow \zeta_{\alpha A}(t) = \alpha\, \zeta_A(t)$

$\Rightarrow S_{\alpha A}(t) = \alpha^{-1} S_A(t)$ $\Leftarrow$ surprising

recall though $S_A(0) = 1/\tau(A)$

Can show:

$$S_A(t) = \frac{1}{R_A(t S_A(t))}$$

$$R_A(g) = \frac{1}{S_A(g R_A(g))}$$

## 11.4.3 Multiplicativity of S

Fix $t$, $\zeta_A$ $\zeta_B$ are inv T-transforms of $t_A$, $t_B$

$$\left.\begin{aligned} E_A &:= (1-A/\zeta_A)^{-1} - 1 - t \\ E_B &:= (1-B/\zeta_B)^{-1} - 1 - t \end{aligned}\right\} \text{Free}$$

$$A/\zeta_A = 1 - (E_A + 1 + t)^{-1}$$

$$\Rightarrow \frac{AB}{\zeta_A \zeta_B} = \left[1-(E_A+1+t)^{-1}\right]\left[1-(E_B+1+t)^{-1}\right]$$

$$= (1+t+E_A)^{-1}\left[\underbrace{(t+E_A)(t+E_B)}_{t^2+t(E_A+E_B)+E_AE_B}\right](1+t+E_B)^{-1}$$

$$t(E_A + E_B) = \frac{t}{1+t}\left[(1+t+E_A)(1+t+E_B) - (1+t)^2 - E_A E_B\right]$$

$$\Rightarrow \frac{AB}{\varsigma_A \varsigma_B} = \frac{t}{1+t} + (1+t+E_A)^{-1}\left[-t(1+t) - E_A E_B \frac{t}{1+t} + t^2 + E_A + E_B\right](1+t+E_B)^{-1}$$

$$= \frac{t}{1+t} + (1+t+E_A)^{-1}\left[-t + \frac{E_A E_B}{1+t}\right](1+t+E_B)^{-1}$$

$$1 - \frac{1+t}{t}\frac{AB}{\varsigma_A \varsigma_B} = (1+t)\,(1+t+E_A)^{-1}\left[1 - \frac{E_A E_B}{t(1+t)}\right](1+t+E_B)^{-1}$$

$$\Rightarrow \left(1 - \frac{1+t}{t}\frac{AB}{\varsigma_A \varsigma_B}\right)^{-1} = \frac{1}{1+t}(1+t+E_A)\underbrace{\left[1 - \frac{E_A E_B}{t(1+t)}\right]^{-1}}_{\sum_{k=0}^{\infty}\left(\frac{E_A E_B}{t(1+t)}\right)^k}(1+t+E_B)$$

$$\Rightarrow \tau\left[\left(1 - \frac{AB}{\varsigma_A \varsigma_B}\frac{1+t}{t}\right)^{-1}\right] = 1+t$$

$$\Rightarrow t_{AB}\Big(\underbrace{\frac{t}{1+t}\varsigma_A \varsigma_B}_{\varsigma_{AB}(t)}\Big) = t \Rightarrow \varsigma_{AB}(t) = \frac{t}{1+t}\varsigma_A \varsigma_B$$

$$\Rightarrow S_{AB}(t) = \left(\frac{t}{1+t}\right)^2 \varsigma_A \varsigma_B = S_A(t)\, S_B(t)$$

## 11.4.4 Subordination

$$\varsigma_{AB}(t) = \frac{\varsigma_A(t)}{S_B(t)}$$

$$t = t_{AB}(\varsigma_{AB}(t)) = t_A(\varsigma_{AB}(t) S_B(t))$$

$$\Rightarrow t_{AB}(\varsigma) = t_A(\varsigma\, S_B(t_{AB}(\varsigma)))$$

Exercise 11.4.1

1) $S_A(t) = \dfrac{t+1}{t\,\varsigma_A(t)}$ $\qquad t = g R_A(g) = zg - 1$

$$S(g\,R(g)) = \frac{zg}{g\,R_A(g)\,\underbrace{S_A(zg-1)}_{z}}$$

$$\Rightarrow \quad R_A(g) = \frac{1}{S(gR(g))}$$

2)

## Chapter 12 : Free Random Matrices

Now we concretely look at large symmetric matrices

$A, B$ free $\Leftrightarrow$ $\tau(p_1(A) q_1(B) p_2(A) \cdots q_n(B)) = 0$

$\tau = \frac{1}{N} \mathrm{Tr}[\cdot]$

$A = U \Lambda U^T$
$B = V \Lambda V^T$ $\Rightarrow$ $\tau(\Lambda_1 O \Lambda_1' O^T \Lambda_2 O \cdots \Lambda_n' O^T) = 0$ $*$

$O = U^T V$

We will show $*$ holds whenever we average over $O$ so long as $\Lambda_i, \Lambda_i'$ are traceless

### 12.1.2 Integration over $O(N)$

Want $\langle \tau(\Lambda_1 O \Lambda_1' O^T \Lambda_2 O \cdots \Lambda_n' O^T) \rangle_O$

First: $I(\vec{i}, \vec{j}, n) := \langle O_{i_1 j_1} O_{i_2 j_2} \cdots O_{i_n j_n} \rangle_O$

Worked out very recently in the general case (Weingarten functions)
2003 – 2008

When $N \to \infty$

$$I(\vec{i}, \vec{j}, n) = N^{n} \sum_{\substack{\text{pairings} \\ \pi}} \delta_{\pi(1)-\pi(2)} \delta_{\pi(3)-\pi(4)} \cdots \delta_{\pi(2n-1)-\pi(2n)} + O(N^{-n-1})$$

$\delta_{\alpha-\beta} := \delta_{i_\alpha i_\beta} \delta_{j_\alpha j_\beta}$

$n=1 \Rightarrow N \langle O_{i_1 j_1} O_{i_2 j_2} \rangle = \delta_{1-2}$ $\leftarrow$ exact

$n=2 \Rightarrow N^2 \langle O_1 O_2 O_3 O_4 \rangle = \delta_{1-2} \delta_{3-4} + \delta_{1-3} \delta_{2-4} + \delta_{1-4} \delta_{2-3}$

$$\Rightarrow \langle \tau(\Lambda_1 O \Lambda_1' O^T) \rangle_O = \frac{1}{N} \sum_{i,j} \Lambda_i \underbrace{\langle O_{ij} \Lambda_j' O^T_{ji} \rangle_O}_{\frac{1}{N} \delta_{ii} \delta_{jj}}$$

$$= \tau(\Lambda) \tau(\Lambda')$$

$$\Rightarrow \lim_{N\to\infty} \langle \tau(\Lambda_1 O \Lambda_1' O^T \Lambda_2 O \Lambda_2' O^T) \rangle_O \quad *$$

$$= \frac{1}{N} \sum_{i,j,k,\ell} \Lambda_i^1 \Lambda_j^{1'} \Lambda_k^2 \Lambda_\ell^{2'} \langle O_{ij} O_{kj} O_{k\ell} O_{i\ell} \rangle$$

$$= \frac{1}{N^3} \sum_{ijk\ell} \Lambda_i^1 \Lambda_j^{1'} \Lambda_k^2 \Lambda_\ell^{2'} (\delta_{ik}^2 + \delta_{j\ell}^2 + \underbrace{\delta_{ik}\delta_{j\ell}}_{\text{subleading}})$$

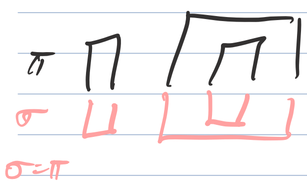
$$= \tau(\Lambda^1\Lambda^2)\tau(\Lambda^{1'})\tau(\Lambda^{2'}) + \tau(\Lambda^1)\tau(\Lambda^2)\tau(\Lambda^{1'}\Lambda^{2'})$$

$\Lambda_1 = \Lambda_2 = \mathbb{1} \Rightarrow \langle \tau(\ldots) \rangle = 1$ but this gives 2 !!!

Subleading terms matter!

Really: $$I(\vec{i}\,\vec{j}\; n) = \sum_{\pi,\sigma} W_n(\pi,\sigma)\, \tilde{\delta}_{1-2} \cdots \tilde{\delta}_{2n-1-2n}$$

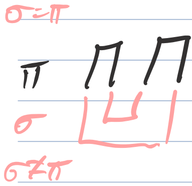
$$\tilde{\delta}_{\alpha\beta} = \delta_{i_{\pi(\alpha)} i_{\pi(\beta)}} \delta_{j_{\sigma(\alpha)} j_{\sigma(\beta)}}$$



\# loops = 3

$W_n$ is matrix on space of perms $\pi, \sigma$

pseudoinverse of $M_n = N^{\ell(\pi,\sigma)}$



\# loops = 1 < 2

$\sigma = \pi \Rightarrow \ell(\pi,\sigma) = n \quad \Rightarrow W \sim N^{-n}$

$\sigma \neq \pi \Rightarrow \ell(\pi,\sigma) < n$

Can show:

$$W_n(\pi,\sigma) = N^{\ell(\pi,\sigma)-2n} \sum_{g=0}^{\infty} \Omega_g(\pi,\sigma) N^{-g} \leftarrow \text{genus?}$$

→ Subleading term in $*$ is $\pi \neq \sigma \quad \ell(\pi,\sigma) = 1 \Rightarrow N^{-3}$

Ex 12.1.1 For $n=2$ there are 3 perms

1. $\ell(\pi_i, \pi_i) = 2$
   $\ell(\pi_i \neq \pi_j) = 1$

2. $M_2 = \begin{pmatrix} N^2 & N & N \\ N & N^2 & N \\ N & N & N^2 \end{pmatrix} \Rightarrow (N^2 - N)\delta + N \mathbb{1}\mathbb{1}^T$

$\to N^2(\mathbb{1} + \frac{1}{N}\mathbb{1}\mathbb{1}^T)$

$\Rightarrow M^{-1} \sim N^{-2}\mathbb{1} - N^{-3}\mathbb{1}\mathbb{1}^T$

$$\Rightarrow W_2 \simeq \begin{pmatrix} N^{-2} & -N^{-3} & -N^{-3} \\ -N^{-3} & N^{-2} & -N^{-3} \\ -N^{-3} & N^{-3} & N^{-2} \end{pmatrix}$$

3.

4. $\langle \tau(OO^T OO^T) \rangle = \frac{1}{N} \sum_{ijkl} \langle O_{ij} O_{kj} O_{kl} O_{il} \rangle$

$$= \frac{1}{N^3} \sum (\delta_{ik}^2 + \delta_{jl}^2 + \delta_{ik}\delta_{jl}) \Lambda\Lambda\Lambda\Lambda$$

$$- \frac{1}{N^4} \underbrace{\sum (\underset{\uparrow\text{ 1st row}}{2\delta_{ik}\delta_{jl}} + \underbrace{\delta_{ik} + \delta_{ik}\delta_{jl}}_{\text{2nd row}} + 1 + \delta_{jl})}_{3\delta_{ik}\delta_{jl} + \delta_{ik} + \delta_{jl} + 1} \Lambda\Lambda\Lambda\Lambda$$

$$= \tau(\Lambda_1\Lambda_2)\tau(\Lambda_1')\tau(\Lambda_2') + \tau(\Lambda_1'\Lambda_2')\tau(\Lambda_1)\tau(\Lambda_2)$$

$$- \tau(\Lambda_1)\tau(\Lambda_2)\tau(\Lambda_3)\tau(\Lambda_4) \cancel{- \frac{3}{N^2}\tau(\Lambda_1\Lambda_2)\tau(\Lambda_1'\Lambda_2')} \quad \text{etc...}$$

$\Rightarrow 1$ to leading order
(can show to subleading as well)

## 12.1.4 Freeness of large matrices

$$\langle \tau(\Lambda_1 O \Lambda_1' O^T \cdots \Lambda_n O \Lambda_n' O^T) \rangle_O$$

$$= \frac{1}{N} \sum_{\vec{i}\,\vec{j}} I(\vec{i}, \vec{j}, n) [\Lambda_1]_{i_{2n} i_1} [\Lambda_2]_{i_2 i_3} \cdots [\Lambda_n']_{j_{2n-1} j_{2n}}$$

$\vec{i}$ & $\vec{j}$ never mix contractions ← like closed strings

$\Rightarrow$ Focus on $\vec{i}$ index

By assumption $\tau(\Lambda_i) = 0 \Rightarrow$ have $\leq \lfloor n/2 \rfloor$ traces
$\Rightarrow \lfloor n/2 \rfloor$ is max power of $N$

Combining $i, j \Rightarrow$ max power is $< N^{-n}$

$$\frac{1}{N} \underset{\uparrow\, O(n!^2)}{\sum_{\pi,\sigma}} \sum_{\vec{i}\,\vec{j}} \underset{\uparrow\, \leq N^{-n}}{W_n} \underbrace{\tilde{\delta}_{1-2} \cdots \tilde{\delta}_{2n-1-2n} [\Lambda_1]_{i_{2n} i_1} [\Lambda_2]_{i_2 i_3} \cdots [\Lambda_n']_{j_{2n-1} j_{2n}}}_{\leq \lfloor \frac{n}{2} \rfloor \text{ free vars over } \vec{i} \text{, over } \vec{j} \;\Rightarrow\; \leq N^{2\lfloor n/2 \rfloor}}$$

N-indep

$$= O(N^{-1-n+2\lfloor n/2\rfloor}) \to 0 \quad \text{as} \quad N \to \infty$$

Can easily be adapted to $U(N)$ $Sp(N)$

## 12.2 R-Transforms from Perturbation Theory

$A + B^R \qquad B^R = O B O^T$

$$g(z) = \langle \tau[(z\mathbb{1} - A - B^R)^{-1}] \rangle_O =: \tau_R[(z\mathbb{1} - A - B^R)^{-1}]$$

expand in $B^R$

$$g(z) = \tau_R(G_A) + \tau_R(G_A B^R G_A) + \cdots$$

free $\Rightarrow$ mixed cumulants vanish

Three types of mixed moments:

B appears n times

$$m_n^{(1)} := \tau_R[GB \cdots BG]$$

$$m_n^2 := \tau_R[BG \cdots GB]$$

$$m_n^3 := \tau_R[BG \cdots BG] = \tau_R[GB \cdots GB]$$

$$m_0^{(1)} = \tau_R[G_A]$$

$$m_0^{(2)} = m_0^{(3)} = 0$$

$$\tilde{M}^{(a)}(u) = \sum_{n=0}^{\infty} m_n^{(a)} u^n$$

care about $g(z) = \tilde{M}^{\prime}(u=1)$

Sum over size $\ell$ of first group $G_A$ belongs to

$$m_n^{(1)} = \sum_{\ell} \kappa_{G_A,\ell} \prod_{\substack{k_1 \cdots k_\ell \\ \sum k_i + \ell = n}} m_{k_1}^{(2)} \cdots m_{k_{\ell-1}}^{(2)} m_{k_\ell}^{(3)}$$

$$m_n^{(2)} = \sum_{\ell} \kappa_{B,\ell} \prod_{\substack{k_1 \cdots k_\ell \\ \sum k_i + \ell = n}} m_{k_1}^{(1)} \cdots m_{k_{\ell-1}}^{(1)} m_{k_\ell}^{(3)}$$

$$m_n^{(3)} = \sum_\ell \kappa_{B\ell} \prod_{\substack{k_1\cdots k_\ell \\ \sum k_i+\ell=n}} m_{k_1}^{(1)} \cdots m_{k_\ell}^{(1)}$$

$$\Rightarrow \tilde{M}^{(1)}(u) = g_A(z) + \sum_{\ell=1} \kappa_{G_A\ell}\, u^\ell\, \tilde{M}^{(2)(\ell-1)} \tilde{M}^{(3)} = g_A + u\tilde{M}^{(3)} R_{G_A}(u\tilde{M}^{(2)})$$

$$\tilde{M}^{(2)}(u) = \sum_{\ell=1} \kappa_{B\ell}\, u^\ell\, \tilde{M}^{(1)(\ell-1)} M^{(3)} = u\tilde{M}^{(3)} R_B(u\tilde{M}^{(1)})$$

$$\tilde{M}^{(3)}(u) = \sum_{\ell=1} \kappa_{B\ell}\, u^\ell\, \tilde{M}^{(1)\ell} = u\tilde{M}^{(1)} R_B(u M^{(1)})$$

$$\Rightarrow g(z) = g_A(z) + g(z) R_B(g(z)) R_{G_A}(g(z) R_B(g(z))^2)$$

Take $B = b\mathbb{1} \Rightarrow R_B = b$

$$g = g_A + bg(z) R_{G_A}(g(z) b^2)$$

$g_A(z-b)$

$$\Rightarrow R_{G_A}(b^2 g_A(z-b)) = \frac{g_A(z-b) - g_A(z)}{b g_A(z-b)}$$

$$\Rightarrow R_{G_A}[b^2 g(z)] = \frac{g(z) - g(z+b)}{b\, g(z)}$$

setting $b = R_B(g)$

$$\Rightarrow R_{G_A}[R_B^2(g) g] \cdot R_B(g)\, g = g - g(z + R_B(g))$$

need $g_A(z) = g(z + R_B\, g(z))$

$$\rightarrow g(z) = g_A(z - R_B(g(z))) \Leftarrow \text{Subordination}$$

# 12.3 CLT

$$M_K = \frac{1}{\sqrt{K}} \sum O_i A_i O_i^T \qquad Tr A_i = 0$$

$$\tau(A_i^{\ell}) = m_\ell$$

$$\Rightarrow R_{A_i} = \sum_{\ell=2}^{\infty} \kappa_\ell z^{\ell-1}$$

$$\Rightarrow R_{M_K} = \sum_{\ell=2}^{\infty} K^{1-\ell/2} \kappa_\ell z^{\ell-1}$$

At finite $K$ $\quad R_{M_K} \approx \underbrace{\sigma^2 z}_{R_X} + \frac{\kappa_3}{\sqrt{K}} \underbrace{z^2}_{R_3} + \cdots$

$\sigma = 1$ WLOG

assume small $\kappa_3$ expansion exists: $\Rightarrow g_{M_K}(z) = g_X(z) + \frac{\kappa_3}{\sqrt{K}} g_3(z) + \cdots$

$$g_X(z) = \frac{1}{2}(z - \sqrt{\Delta}) \qquad \Delta = z^2 - 4$$

$$\underbrace{R_{M_K}(g_{M_K}(z))}_{z(g)-1/g} = R_X\left[g_X(z) + \frac{\kappa_3}{\sqrt{K}} g_3(z)\right] + R_3(g_X(z))$$

$$\Rightarrow z - \frac{1}{g_X + \frac{\kappa_3}{\sqrt{K}} g_3} = g_X + \frac{\kappa_3}{\sqrt{K}} \frac{g_3}{g_X^2} = g_X + \frac{\kappa_3}{\sqrt{K}} g_3(z) + R_3(g_X(z))$$

$$\Rightarrow \frac{g_3(1-g_X^2)}{g_X^2} = R_3(g_X)$$

$$\Rightarrow g_3 = -\frac{1}{2}\left[1 - \frac{z}{\sqrt{z^2-4}}\right][g_X(z)]^2$$

$$= -g_X'(z)\, g_X(z)^2$$

$$= -\frac{1}{4}\left(1 - \frac{z\sqrt{\Delta}}{\Delta}\right)\left(z^2 - 2 - z\sqrt{\Delta}\right)$$

$z = \lambda + i0$

$\Rightarrow z^2 = \lambda^2 + 2i\lambda 0$

$$\Rightarrow z^2-4 = \lambda^2-4+2i\lambda 0^+$$

$$\frac{\operatorname{Im}\rho}{\pi} = \underbrace{\frac{\kappa_3}{2\pi\sqrt{\kappa}}\,\frac{\lambda(\lambda^2-3)}{\sqrt{4-\lambda^2}}}_{\delta\rho}$$

$$\int_{-2}^{2}\lambda^3\,\delta\rho(\lambda) = \frac{\kappa_3}{\sqrt{\kappa}} \checkmark$$

For $\kappa_3=0$ $\quad \kappa_4\neq 0$

$$R_{M_\kappa}(z) = \underset{1}{\sigma^2} z + \frac{\kappa_4}{\kappa}z^3 \qquad g_{M_\kappa} = g_X + \frac{\kappa_4}{\kappa}g_4 + \cdots$$

$$z - \frac{1}{g_X+\frac{\kappa_4}{\kappa}g_4} \approx g_X + \frac{\kappa_4}{\kappa}\frac{g_4}{g_X^2} = g_X + \frac{\kappa_4}{\kappa}g_4 + \frac{\kappa_4}{\kappa}(g_X)^3$$

$$\Rightarrow g_4\frac{(1-g_X^2)}{g_X^2} = g_X^3$$

$$\Rightarrow \delta\rho = \frac{\kappa_4}{2\pi\kappa}\,\frac{\lambda^4-4\lambda^2+2}{\sqrt{4-\lambda^2}}$$

```
In[642]:= 1/2 (-1 + z/Sqrt[-4 + z^2]) gX^3 // Simplify
Out[642]= (z - Sqrt[-4 + z^2])^4 / (16 Sqrt[-4 + z^2])
In[643]:= (z - Sqrt[-4 + z^2]) / (16 Sqrt[-4 + z^2]) // Simplify
In[645]:= 1/16 (-1 + z/Sqrt[-4 + z^2]) (z - Sqrt[-4 + z^2])^3 // Expand
Out[645]= z - z^3/2 + z^4/(16 Sqrt[-4 + z^2]) - 1/4 Sqrt[-4 + z^2] + 7/16 z^2 Sqrt[-4 + z^2]
In[646]:= Assuming[-2 < λ < 2 && ϵ > 0,
  Simplify[Im[z^4/(16 Sqrt[-4 + z^2]) - 1/4 Sqrt[-4 + z^2] + 7/16 z^2 Sqrt[-4 + z^2] /. z → λ + I ϵ]]]
Out[646]= 1/2 Im[(2 + ϵ^4 - 4 i ϵ^3 λ - 4 λ^2 + λ^4 + ϵ^2 (4 - 6 λ^2) + 4 i ϵ λ (-2 + λ^2)) / Sqrt[-4 + (i ϵ + λ)^2]]
```

$$\delta\rho_n(\lambda) = \frac{\kappa_n}{\pi\kappa^{n/2-1}}\,\frac{T_n(\lambda/2)}{\sqrt{4-\lambda^2}}$$

## 12.4 Finite Free Convolutions

$$p(z) = \prod_i(z-\lambda_i) = \det(z\mathbb{1}-A)$$
$$= \sum(-1)^k a_k z^{N-k}$$

$$a_k = \sum_{i\in T_k}\lambda_{i_1}\cdots\lambda_{i_k}$$

ordered k-tuple
$i_1 < i_2 \cdots < i_N$

$$\mu(\{\lambda_i\}) = \frac{a_1}{N} \qquad \hat\sigma^2(\{\lambda_i\}) = \frac{1}{N-1}\sum_i(\lambda_i-\mu)^2 = \frac{1}{N-1}\sum\lambda_i^2 - \frac{N}{N-1}\mu^2$$

$$= \frac{1}{N-1}(a_1)^2 - \frac{2}{N-1}a_2 - \frac{N}{N-1}\left(\frac{a_1}{N}\right)^2$$

$$= \left[\frac{N-1}{(N-1)N}\right] a_1^2 - \frac{2}{N-1} a_2$$

$$= \frac{a_1^2}{N} - \frac{2a_2}{N-1}$$

$p(z)$ will often be $\mathbb{E}[\det(z\mathbb{1}-A)]$

$\Rightarrow$ think of $\lambda_i$ as deterministic

$\lambda_i$ indep $\Rightarrow \mathbb{E}\, p(z) = (z - \mathbb{E}\mu_i)^N$

$\Rightarrow \lambda_i = \mathbb{E}(\mu) \ \forall i$

$$p_{M+\alpha\mathbb{1}}(z) = p_M(z-\alpha)$$

$$p_{\alpha M}(z) = \alpha^N p_M(\alpha^{-1}z) \Rightarrow \boxed{\tilde{a}_k = \alpha^k a_k}$$

$$p_{M^{-1}} = \frac{(-z)^N}{a_N} p_M(1/z) \Rightarrow \boxed{\tilde{a}_k = \frac{a_{N-k}}{a_N}}$$

$$= \frac{z^N}{\lambda_1 \cdots \lambda_N}\left(\frac{1}{z}-\lambda_1\right)\cdots\left(1/z - \lambda_N\right)$$

$$= \left(\tfrac{1}{\lambda_1} - z\right)\cdots\left(1/\lambda_N - z\right)$$

$$p(z) =: \hat{p}(\partial_z) z^N$$

$$\hat{a}_k = (-1)^N \frac{k!}{N!} a_{N-k}$$

coeff of $(-1)^{N-k}\partial_z^{N-k}$ yielding $(-1)^k z^k$

## 12.4.2 Finite free addition:

$p_1, p_2$ monic

$$p_1 \boxplus p_2(z) = \left\langle \det[z\mathbb{1} - \Lambda_1 - O\Lambda_2 O^T]\right\rangle_O$$ "Free additive convolution"

$O \in O(N)$ or $U(N)$ or even $S_N$

result is the same!

Will show soon: $p_1 \boxplus p_2 = \hat{p}_1(\partial_z) p_2(z) = \hat{p}_2(\partial_z) p_1(z) = \hat{p}_1(\partial_z)\hat{p}_2(\partial_z) z^N$

Turns out $p_1 \boxplus p_2$ still has real roots!

$\Rightarrow \boxplus$ is bilinear in the coeffs

$\Rightarrow$ IF $p_1$ $p_2$ are polys of indep random matrices

$$\mathbb{E}[p_1 \boxplus p_2] = \mathbb{E}[p_1] \boxplus \mathbb{E}[p_2$$

$$a_k^s = \sum_{i+j=k} \frac{(N-i)!(N-j)!}{N!\,(N-k)!} a_i^{(1)} a_j^{(2)}$$

$a_0^s = 1$
$a_1^s = a_1^{(1)} + a_1^{(2)}$
$a_2^s = a_2^{(1)} + a_2^{(2)} + \frac{N-1}{N} a_1^{(1)} a_1^{(2)}$

$\Rightarrow$ sample mean & variance adds under $\boxplus$

let $p_0(z) = z^N \leftarrow a_0 = 1 \quad a_i = 0 \quad \forall i > 0 \quad \Rightarrow \quad p \boxplus p_0 = p(z)$

let $p_\mu(z) = (z-\mu)^N \Rightarrow p \boxplus p_\mu = p(z-\mu)$

Can show Hermite polynomials are stable under $\boxplus$

$H_N \boxplus H_N = 2^{N/2} H_N(2^{-N/2} z)$ because $\hat{H}_N(\partial_z) = e^{-\partial_z^2/2}$ by the differential operator rep'n of $H_N$

$\widehat{H \boxplus H} = \hat{H} \cdot \hat{H} = e^{-\partial_z^2}$

Wishart rank 1: $M = xx^T \qquad x \sim N(0, \mathbb{1} \cdot N)$

$\mathbb{E}$ char poly $\rightsquigarrow p(z) = z^{N-1}(z-N) = (1-\partial_z) z^N$

$\Rightarrow \hat{p}(\partial_z) = 1 - \partial_z$

For Wishart of param $T$ $\sum_t x_t x_t^T$ is the free sum of $T$ such rank 1 projectors

$\Rightarrow \hat{p}_T(\partial_z) = (1-\partial_z)^T$

$\Rightarrow p_T(z) = (1-\partial_z) z^N$

## 12.4.3 Finite R-Transform

$\log \hat{p}(\partial_z)$ is additive under $\boxplus$

Define all $\hat{p}(\partial_z)$ mod $\partial_z^{N+1} \Rightarrow$ finite dim ring

$\hat{p}(\partial_z) = 1 + O(\partial_z)$

$\Rightarrow$ define $\log \hat{p}(u)$ as a formal series truncated beyond $u^N$

$L(u) := -\log \hat{p}(u) \mod u^{N+1}$

Eg 1 $\quad p_1 = (z-1)^N = \sum_k (-1)^k \binom{N}{k} z^{N-k} = \sum_k \frac{(-\partial_z)^k}{k!} z^N = \exp(-\partial_z) z^N$

$\Rightarrow \hat{p}_1(u) = \exp(-u) \mod u^{N+1} \Rightarrow L_1(u) = u$

**Eg 2** For Wigner $\mathbb{E}$ char poly is

$$p_X(z) = N^{-N/2} H_N(\sqrt{N} z) = \exp\left[-\frac{1}{2N}\partial_z^2\right] z^N$$

$$\Rightarrow L_X(u) = \frac{u^2}{2N}$$

**Eg 3** For Wishart $\mathbb{E}$ char poly is normalized Laguerre:

$$p_W(z) = \left(1 - \frac{1}{T}\frac{\partial}{\partial z}\right)^T z^N \qquad T = N/q$$

$$\Rightarrow L_W(u) = -\frac{N}{q}\log\left(1 - \frac{qu}{N}\right) \mod u^{N+1}$$

In these three cases:

$$L'(u) = [R(u/N)] \mod u^{N+1}$$

Generally:

$$\lim_{N\to\infty} L'(Nu) = R(u)$$

## 12.4.4 Finite free product

$$p_1 \boxtimes p_2 := \left\langle \det[z\mathbb{1} - \Lambda_1 O \Lambda_2 O^T] \right\rangle_O$$

$O$ over $O(N)$, $U(N)$, $S_N$ - doesn't matter

Will show:

$$a_k^{(m)} = \binom{N}{k}^{-1} a_k^{(1)} a_k^{(2)}$$

$$p_{\alpha\mathbb{1}} = (z-\alpha)^N \Rightarrow a_k = \binom{N}{k}\alpha^k$$

$$\Rightarrow p \boxtimes p_{\alpha\mathbb{1}} \text{ has } a_k \to \alpha^k a_k$$

ie $\lambda_j \to \alpha\lambda_j$

$$a_1^{(m)} = \frac{1}{N} a_1^{(1)} a_1^{(2)} \Rightarrow \mu^{m} = \mu^{(1)} \cdot \mu^{(2)}$$

Assume $\mu^{(1)}, \mu^{(2)} = 1 \Rightarrow \mu^{(m)} = 1$, $\sigma_i^2 = N - \frac{2a_i}{N-1}$ $\Rightarrow \frac{\sigma_{(1)}^2 \sigma_{(2)}^2}{N} = N + \frac{4 a_2^{(1)} a_2^{(2)}}{(N-1)^2 N} - \frac{2(a_2^{(1)} + a_2^{(2)})}{(N-1)}$

$$\sigma_{(m)}^2 = N - \frac{2a_2^{(m)}}{N-1} = N - \frac{4 a_2^{(1)} a_2^{(2)}}{N(N-1)^2} = \sigma_{(1)}^2 + \sigma_{(2)}^2 - \frac{\sigma_{(1)}^2 \sigma_{(2)}^2}{N}$$

Ex 12.4.1 $\quad p(z) = z^M (z-1)^M$

a) $p(z) = z^M \sum_{k=1}^{M} (-1)^k \binom{M}{k} z^{M-k} \Rightarrow a_k = \binom{M}{k} (-1)^M$

b) $p_m = p \boxtimes p$ has $a_k^{(m)} = \begin{cases} \binom{M}{k}^2 \binom{2M}{k}^{-1} & k \le M \\ 0 & \text{else} \end{cases}$

c) $p_m$ still has zero of mult $M \Rightarrow q(z) = z^{-M} p_m(z)$ poly w/ $M$ roots

d) Average root is $a_1 = \frac{M^2}{N} = \frac{M}{2}$

e) $z^2(z-1)^2 \Rightarrow p_m = z^2 \cdot \left(z^2 + \frac{M}{2} z + \frac{(M(M-1))^2}{2^2} \frac{2}{2M(2M-1)}\right)$

$$= z^2\left(z^2 + z + \frac{2}{12}\right)$$

$$\Rightarrow r_\pm = \frac{1}{2} \pm \frac{\sqrt{1 - 2/3}}{2} = \frac{1}{2} \pm \frac{1}{\sqrt{12}}$$

f) $z^4(z-1)^4 \Rightarrow p_m = z^4\left(z^4 + \frac{M}{2} z^3 + \frac{(M(M-1))^2}{4} \frac{2}{2M(2M-1)} z^2 + M(M-1)(M-2) \dots\right)$

pointless

D.4.5 Derivation of Results

$$p_S(z) := p_1 \boxplus p_2 = \underbrace{\frac{1}{N!} \sum_{\text{perms } \sigma} \prod_{i=1}^{N} \left(z - \lambda_i^{(1)} - \lambda_{\sigma(i)}^{(2)}\right)}_{\det(z\mathbb{1} - \Lambda^1 - O \Lambda^2 O^T)} \qquad O \in S_N$$

$$a_1^{(S)} = \frac{1}{N!} \sum_{\sigma} \sum_{i} \left(\lambda_i^{(1)} + \lambda_{\sigma(i)}^{(2)}\right) = \sum_i \left(\lambda_i^{(1)} + \lambda_i^{(2)}\right) = a_1^{(1)} + a_1^{(2)}$$

For other $a_k^{(S)}$:

$$\left(z - \lambda_1^{(1)} - \lambda_{\sigma(1)}^{(2)}\right) \cdots \left(z - \lambda_N^{(1)} - \lambda_{\sigma(N)}^{(2)}\right)$$

choose $z$ $N-k$ times, $\lambda^{(1)}$ $i$ times $\lambda^2$ $k-i$ times

Once averaged over $\sigma \in S_N$ the product of the $\lambda^{(1)} \lambda^{(2)}$ must be completely symmetric in $\lambda^{(1)}, \lambda^{(2)}$ and therefore $\propto a_j^{(1)} a_{k-j}^{(2)}$

$$a_k^{(s)} = \sum_{j=0}^{k} \underbrace{C(j,k,N)}_{\text{TBD}} a_j^{(1)} a_{k-j}^{(2)}$$

$\Rightarrow$

$$\Lambda_2 = \mathbb{1} \Rightarrow p_{\mathbb{1}}(z) = (z-1) = \sum_k (-1)^k \underbrace{\binom{N}{k}}_{a_k} z^{N-k}$$

$$p_s = p \boxplus p_{\mathbb{1}} = p(z-1) = \sum_k (-1)^k a_k (z-1)^{N-k}$$

$$\Rightarrow a_k^{(s)} = \sum_j \binom{N-j}{N-k} a_j$$

$$\Rightarrow a_k^{(s)} = \sum_k (-1)^k z^{(N-k)} \sum_{j=0}^{k} \binom{N-j}{N-k} a_j$$

$$= \sum_k (-1)^k z^{N-k} \sum_{j=0}^{k} C(j,k,N) a_j^{(1)} \underbrace{a_{k-j}^{(2)}}_{\binom{N}{k-j}} \Rightarrow C(j,k,N) = \binom{N-j}{N-k}\binom{N}{k-j}^{-1}$$

$$= \frac{(N-j)!\,(N-k+j)!\,\cancel{(k-j)!}}{N!\,(N-k)!\,\cancel{(k-j)!}}$$

Now see next exercise

Now for $O(N)$, $U(N)$

Expand det in powers of $z$, $\lambda^{(1)}$

After averaging $\lambda_j^{(1)}$ appearances must be perm-invariant

$$\Rightarrow a_k^{(s)} = \sum_{j=1}^{k} C(j,k,N,\{\lambda^{(2)}\})\, a_j^{(1)}$$

By dimensional analysis + $S_N$-symm on $\lambda^{(2)}$ we get

$$= \sum_{j=1}^{k} \tilde{C}(j,k,N)\, a_j^{(1)} a_{k-j}^{(2)}$$

Because $p_s = p \boxplus p_{\mathbb{1}}$ must be the same $\forall p$ regardless of the group average, we get $\tilde{C} = C$

Now for $\boxtimes$:

$$p_m(z) = \frac{1}{N!}\sum_{\sigma}\prod_{j=1}^{N}\left(z - \lambda_j^{(1)}\lambda_{\sigma(j)}^{(2)}\right) =: \frac{1}{N!}\sum_{\sigma} p_\sigma(z)$$

$$a_k^{\sigma} = \sum_{j\in T_k} \lambda_{j_1}^{(1)}\cdots\lambda_{j_k}^{(1)}\,\lambda_{\sigma(j_1)}^{(2)}\cdots\lambda_{\sigma(j_k)}^{(2)}$$

ordered k-tuple

After $\sigma$-avg we get $a_k^{(s)} \propto a_k^{(1)} a_k^{(2)}$

proportionality const must be $\left[\sum_{j\in T_k}\right]^{-1} = \binom{N}{k}^{-1}$

or by requiring $p \boxtimes p_{I} = p$

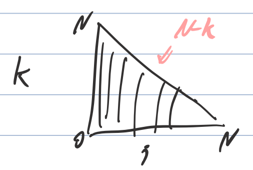
Ex 12.4.2

a) $$\hat{p}(\partial_z) z^N = \sum_k \frac{(N-k)!}{N!} a_k (-1)^k \underbrace{\partial_z^k z^N}_{\frac{N!}{(N-k)!} z^{N-k}} = p(z)$$

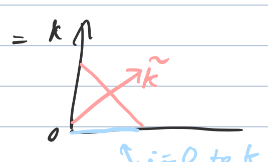
b) $p_1 \boxplus p_2 = \hat{p}_1(\partial_z) p_2$

$$\hat{p}_1(\partial_z) p_2(z) = \sum_{j=0}^{N} (-1)^k \frac{(N-j)!}{N!} a_j^{(1)} \partial_z^j \sum_{k=0}^{N-j} (-1)^j a_k^{(2)} z^{N-k}$$

$$= \sum_{j+k\le N} (-1)^{j+k} \frac{(N-j)!}{N!} \frac{(N-k)!}{(N-j-k)!} a_j^{(1)} a_k^{(2)} z^{N-j-k}$$




$$= \sum_{k=0}^{N}\sum_{j=0}^{k} (-1)^k \underbrace{\frac{(N-j)!}{N!} \frac{(N-k+j)!}{(N-k)!} a_j^{(1)} a_{k-j}^{(2)}}_{12.90 \text{ defining } \boxplus} z^{N-k}$$



## 12.5 Freeness for $2\times2$ Matrices

$N=1 \Rightarrow$ $1\times1$ matrices commute $\Rightarrow$ only constants are free

$N=2 \Rightarrow$ Def $\tau = \frac{1}{2}\mathbb{E}\,\mathrm{Tr}\,A$

Take $A$ with evals deterministic
evecs random

$$A = a\mathbb{1} + \sigma\, O \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} O^T$$

$$O = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \Rightarrow A = a\mathbb{1} + \sigma \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

### 12.5.1

$\Rightarrow$ traceless polynomials take the form:

$$p_k = a_k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad q_k = b_k \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

$$\tau\Big[\prod_k p_k(A)\, q_k(B)\Big] = \frac{1}{2}\prod_k a_k b_k\, \mathrm{Tr}\, \underbrace{\mathbb{E}\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}^n}_{0 \text{ by symmetry } \Rightarrow \text{ Free!}}$$

**Ex 12.5.1** a) $A_1$ has $\lambda = \pm\sigma$

$$\Rightarrow \frac{1}{2}\mathrm{Tr}\frac{1}{z-A} = \frac{1}{2}\left[\frac{1}{z-\sigma} + \frac{1}{z+\sigma}\right] \Rightarrow \frac{z}{z^2-\sigma^2}$$

$$\Rightarrow z = \frac{1+\sqrt{1+4g^2\sigma^2}}{2g} \Rightarrow R = \frac{-1+\sqrt{1+4g^2\sigma^2}}{2g}$$

b)

```
In[792]:= (z /. Solve[g == z/(z^2 - σ^2), z][[2]]) - 1/g // Simplify
Out[792]= (-1 + Sqrt[1 + 4 g^2 σ^2])/(2 g)

In[800]:= Solve[z == (-1 + Sqrt[1 + 4 g^2 σ1^2])/(2 g) + (-1 + Sqrt[1 + 4 g^2 σ2^2])/(2 g) + 1/g, g][[2]] // Simplify
Solve: There may be values of the parameters for which some or all solutions are not valid.
Out[800]= {g -> ± z/Sqrt[z^4 + (σ1^2 - σ2^2)^2 - 2 z^2 (σ1^2 + σ2^2)]}
```

c)

```
In[807]:= -Det[{{σ1 + σ2 Cos[2 θ], σ2 Sin[2 θ]}, {σ2 Sin[2 θ], -σ1 - σ2 Cos[2 θ]}}] // FullSimplify
Out[807]= σ1^2 + σ2^2 + 2 σ1 σ2 Cos[2 θ]
```

d)

$$\text{In[813]:=}\quad \frac{1}{2}\left(\frac{1}{z-\text{Sqrt}\left[\sigma 1^2+\sigma 2^2+2\,\sigma 1\,\sigma 2\,\text{Cos}[2\,\theta]\right]}+\frac{1}{z+\text{Sqrt}\left[\sigma 1^2+\sigma 2^2+2\,\sigma 1\,\sigma 2\,\text{Cos}[2\,\theta]\right]}\right)\ \text{// FullSimplify}$$

$$\text{Out[813]=}\quad -\frac{z}{-z^2+\sigma 1^2+\sigma 2^2+2\,\sigma 1\,\sigma 2\,\text{Cos}[2\,\theta]}$$

$$\text{In[821]:=}\quad \frac{1}{2\,\pi}\,\text{Integrate}\left[\frac{z}{z^2-\left(\sigma 1^2+\sigma 2^2+2\,\sigma 1\,\sigma 2\,\text{Cos}[2\,\theta]\right)},\ \{\theta,0,2\,\pi\}\right]$$

$$\text{In[826]:=}\quad -\frac{1}{2\,\pi}\,z\left(-\left(\left(2\,\pi\left(-z^2+\sigma 1^2+\sigma 2^2+\sqrt{-4\,\sigma 1^2\,\sigma 2^2+\left(-z^2+\sigma 1^2+\sigma 2^2\right)^2}\right)\right)\Big/\left(-4\,\sigma 1^2\,\sigma 2^2+\left(-z^2+\sigma 1^2+\sigma 2^2\right)^2-z^2\sqrt{\left(z^4-2\,z^2\,\sigma 1^2+\sigma 1^4-2\,z^2\,\sigma 2^2-2\,\sigma 1^2\,\sigma 2^2+\sigma 2^4\right)}+\sigma 1^2\sqrt{\left(z^4-2\,z^2\,\sigma 1^2+\sigma 1^4-2\,z^2\,\sigma 2^2-2\,\sigma 1^2\,\sigma 2^2+\sigma 2^4\right)}+\sigma 2^2\sqrt{\left(z^4-2\,z^2\,\sigma 1^2+\sigma 1^4-2\,z^2\,\sigma 2^2-2\,\sigma 1^2\,\sigma 2^2+\sigma 2^4\right)}\right)\right)\right)\ \text{// FullSimplify}$$

$$\text{Out[826]=}\quad \frac{z}{\sqrt{(z-\sigma 1-\sigma 2)\,(z+\sigma 1-\sigma 2)\,(z-\sigma 1+\sigma 2)\,(z+\sigma 1+\sigma 2)}}$$

as before

## Ex 12.5.2

$$A_1 = O\begin{pmatrix}0 & 0\\ 0 & a_1\end{pmatrix}O^T$$

a) $$t_A = \tau\left[\frac{1}{1-\varsigma^{-1}A}\right]-1 = \frac{1}{2}\left(\frac{1}{1}+\frac{1}{1-\varsigma^{-1}a_1}\right)-1 = \frac{1}{2}\frac{a_1}{\varsigma-a_1}$$

$$\Rightarrow \varsigma_A(t) = \frac{a_1(1+2t)}{2t} \Rightarrow S_A(t) = \frac{2}{a_1}\frac{1+t}{1+2t}$$

b) $$S_{A_1A_2} = \frac{2}{a_1a_2}\frac{(1+t)^2}{(1+2t)^2} \Rightarrow \varsigma_{A_1A_2} = \frac{a_1a_2}{2}\frac{(1+2t)^2}{(1+t)t}$$

$$\Rightarrow t = -\frac{1}{2}-\frac{1}{2}\sqrt{\frac{\varsigma}{\varsigma-a_1a_2}}$$

$$\Rightarrow \frac{1}{z}(t+1) = \frac{1}{2z}-\frac{1}{2}\frac{1}{\sqrt{(z-a_1a_2)z}}$$

$\varsigma = z$

$$\Rightarrow p(\lambda) = \frac{1}{2}\delta(\lambda)+\frac{1}{2\pi}\frac{1}{\sqrt{\lambda(a_1a_2-\lambda)}} \qquad 0\le\lambda\le a_1a_2$$

shifted arcsine

c) $$A_1 = \begin{pmatrix}0 & 0\\ 0 & a_1\end{pmatrix}$$

$$A_2 = O\begin{pmatrix}0 & 0\\ 0 & a_2\end{pmatrix}O^T = a_2\begin{pmatrix}\sin^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \cos^2\theta\end{pmatrix}$$

$$\sqrt{A_1}\,A_2\,\sqrt{A_1} = a_1a_2\begin{pmatrix}0 & 0\\ 0 & \cos^2\theta\end{pmatrix}$$

$$\Rightarrow \theta = \arccos\sqrt{\frac{\lambda}{a_1a_2}} \qquad \theta\sim \mathrm{Unif}\left(0,\pi/2\right) \qquad p(\theta) = (2\pi)^{-1}$$

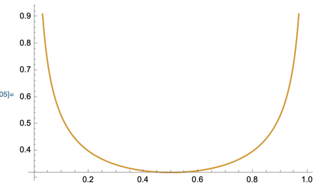
no double cover

$$p(\lambda) = \frac{d\theta}{d\lambda} = \frac{1}{\pi\cdot\cos\theta\sin\theta\, a_1a_2}$$

$$\frac{d\lambda}{d\theta} = 2\cos\theta\sin\theta \quad a_1 a_2$$

In[905]:= Plot[{(1/π) Csc[2 ArcSec[1/Sqrt[λ]]], (1/(2π)) 1/Sqrt[λ (1 - λ)]}, {λ, 0, 1}]

Out[905]=




## 12.5.3 Pairwise Freeness $\neq$ Joint Freeness

Take $A, B, C$ traceless $\sigma^2 = 1$, deterministic $\lambda_i$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B = O_1 A O_1^T \qquad C = O_2 B O_2^T$$

$$ABC = \begin{pmatrix} \cos 2(\theta-\varphi) & \sin 2(\theta-\varphi) \\ \sin 2(\theta-\varphi) & -\cos 2(\theta-\varphi) \end{pmatrix} \Rightarrow \tau(ABC) = 0$$

$\kappa_1 = 0$

$\kappa_3 = 0$

$\kappa_6 = 1$

$(ABC)^2 = \mathbb{1} \Leftarrow \tau \neq 0 \Rightarrow$ not free as a collection

Can show $\kappa_6(A+B, A+B, C, A+B, A+B, C) = 4 \neq 0$

$\Rightarrow$ $A+B$ not free of $C$

$\Rightarrow$ Do not satisfy free CLT

$\Rightarrow$ Sums of $A_{2\times 2}$ w/ $\lambda_i$ deterministic do not yield Wigner $\rho(\lambda)$

## Ex 12.5.3

a) $P(\{\lambda_i\}) = \prod_{k<l} |\lambda_k - \lambda_l| \exp\left[-\frac{N}{2}\sum_i V(\lambda_i)\right]$

$$\Rightarrow P(\lambda_1, \lambda_2) = |\lambda_1 - \lambda_2|\, e^{-\frac{\lambda_1^2}{2} - \frac{\lambda_2^2}{2}}$$

b)

```
In[936]:= (Integrate[Exp[-λ1^2/2 - λ2^2/2] (λ1 - λ2), {λ1, λ2, Infinity}] +
    Integrate[Exp[-λ1^2/2 - λ2^2/2] (λ2 - λ1), {λ1, -Infinity, λ2}]) // FullSimplify
Out[936]= e^(-λ2^2) (2 + e^(λ2^2/2) Sqrt[2 π] λ2 Erf[λ2/Sqrt[2]])

In[937]:= Integrate[Exp[-λ1^2/2 - λ2^2/2] Abs[λ1 - λ2], {λ1, -Infinity, Infinity},
    {λ2, -Infinity, Infinity}]
Out[937]= 4 Sqrt[π]
```

$$\Rightarrow \frac{e^{-\lambda^2}}{4\sqrt{\pi}}\left(2 + e^{\lambda^2/2}\sqrt{2\pi}\,\lambda\, \mathrm{erf}\left(\frac{\lambda}{\sqrt{2}}\right)\right)$$

c) $\int d\lambda_1 \, (\lambda_1 - \lambda_2)^2 \, e^{-\lambda_1^2 - \lambda_2^2}$

d) $= \frac{e^{-\lambda^2}}{\sqrt{\pi}}\left(\lambda^2 + \frac{1}{2}\right)$ ← not semicircle

# Chapter 13: The Replica Method

Let's now recover all major results so far using replicas!

$$\langle \log Z \rangle = \lim_{n\to 0} \frac{\langle Z^n \rangle - 1}{n}$$

We can usually only compute $\lim_{n\to 0} \lim_{N\to\infty} Z_N^n$

But we want $\lim_{N\to\infty} \lim_{n\to 0} Z_N^n$

## 13.1 Stieltjes Transform

### 13.1.1 General setup:

$$\mathbb{E}\, g_A(z) = \frac{1}{N} \mathbb{E}\left[\sum_k \frac{1}{z-\lambda_k}\right] = \frac{1}{N} \mathbb{E}\left[\frac{d}{dz} \log \underbrace{\det(z\mathbb{1}-A)}_{Z^{1/2}}\right]$$

$$= \frac{1}{N} \frac{d}{dz} \mathbb{E}_A \frac{1}{2} \log Z$$

$$= \frac{d}{dz} \mathbb{E}_A \frac{Z^n - 1}{2Nn}$$

$$\mathbb{E}\, Z^n = \int \frac{d^{Nn}\psi_\alpha}{(2\pi)^{Nn/2}} \mathbb{E}_A \exp\left[-\frac{1}{2}\psi_\alpha^T (z\mathbb{1}-A)\psi_\alpha\right]$$

$$g := \lim_{N\to\infty} g_A = \lim_{N\to\infty} \lim_{n\to 0} [\cdots]$$

We can use replicas for $\lim_{n\to 0} \lim_{N\to\infty} [\cdots]$

### 13.1.2 Wigner Case

$A = X$ symmetric Gaussian

$$\int \frac{d^{Nn}\psi_\alpha^i}{(2\pi)^{Nn/2}} e^{-\frac{z}{2}\psi_\alpha^T\psi_\alpha} \mathop{\mathbb{E}}_{\substack{X_{i\neq j} \\ \sim N(0,\frac{\sigma^2}{N})}} \exp\left[\psi_\alpha^i X_{i<j} \psi_\alpha^j\right] \mathop{\mathbb{E}}_{\substack{X_{ii} \\ \sim N(0,\frac{2\sigma^2}{N})}} \exp\left[\frac{1}{2}\psi_\alpha^i X_{ii} \psi_\alpha^i\right]$$

$$= \int \frac{d^{Nn}\psi_\alpha^i}{(2\pi)^{N/2}} e^{-z\psi_\alpha^T\psi_\alpha} \exp\left[\frac{\sigma^2}{2N}\sum_{i<j}\left(\sum_\alpha \psi_\alpha^i \psi_\alpha^j\right)^2 + \frac{\sigma^2}{4N}\sum_i \left(\sum_\alpha \psi_\alpha^i \psi_\alpha^i\right)^2\right]$$

$$= \int [\cdots]\, e^{-z\psi^T\psi} \exp\Bigg[\underbrace{\frac{\sigma^2}{4N}\sum_{ij}\Big(\sum_\alpha \psi_\alpha^i \psi_\alpha^j\Big)^2}_{\psi_\alpha^i \psi_\beta^i \psi_\alpha^j \psi_\beta^j}\Bigg] \leftarrow \text{Factorizes over } i,j$$

$$\frac{\sigma^2}{4N}\sum_{\alpha,\beta}\Big(\sum_i \psi_\alpha^i \psi_\beta^i\Big)^2 \leftarrow \text{Factorizes in } \alpha,\beta \qquad \text{(allows us to define } q_{ab}\text{)}$$

$$= \frac{\sigma^2}{2N}\sum_{\alpha<\beta}\Big(\underbrace{\sum_i \psi_\alpha^i \psi_\beta^i}_{q_{\alpha\beta}}\Big)^2 + \frac{\sigma^2}{N}\sum_\alpha\Big(\underbrace{\sum_i \frac{\psi_\alpha^i \psi_\alpha^i}{2}}_{q_{\alpha\alpha}}\Big)^2$$

$$= \int dq_{\alpha\beta}\, d\vec{\psi}_\alpha \exp\Big[-\frac{N\,\mathrm{Tr}\,q^2}{4\sigma^2} - \frac{1}{2}\sum_{\alpha,\beta,i}(z\delta_{\alpha\beta} - q_a q_b)\,\psi_\alpha^i \psi_\beta^i\Big] \leftarrow \text{Factors over } i$$

$$= \int dq_{\alpha\beta} \exp\Big[\underbrace{-\frac{N\,\mathrm{Tr}\,q^2}{4\sigma^2} - \frac{N}{2}\,\mathrm{Tr}\log(z\mathbb{1} - q)}_{-\frac{N}{2}F(q)}\Big]$$

$$= \int \underset{\uparrow\ \text{eigs of } q}{dq_\alpha} \exp\Big\{-N\Big[\sum_\alpha\Big(\frac{q_\alpha^2}{2\sigma^2} + \log(z - q_\alpha)\Big) - \underbrace{\frac{1}{N}\sum_{\alpha\neq\beta}\log q_\alpha - q_\beta}_{\text{Vandermonde}}\Big]\Big\}$$

$$F(q_\alpha) = \sum_\alpha\Big[\frac{q_\alpha^2}{\sigma^2} + \log(z - q_\alpha)\Big]$$

$$\Rightarrow \frac{q_\alpha}{\sigma^2} - \frac{1}{z - q_\alpha} - \cancel{\frac{1}{N}\sum_{\beta\neq\alpha}\frac{2}{q_\alpha - q_\beta}} = 0 \qquad \mathcal{O}(n/N) \to 0$$

$$\Rightarrow q_\alpha = \frac{\sigma^2}{z - q_\alpha} \qquad \text{S.C. Eq} \qquad \text{same } \forall \alpha \to \text{factor of } n \text{ out front}$$

$$\Rightarrow \mathbb{E}[z^n] \simeq \exp\Big[-\frac{Nn}{2}F_1(z, q^*(z))\Big]$$

$$\Rightarrow \lim_{n\to 0}\lim_{N\to\infty} \frac{\mathbb{E}[z^n] - 1}{Nn} = -\frac{1}{2}F_1(z, q^*(z))$$

$$\Rightarrow g_X(z) = \frac{d}{dz}F_1(z, q^*(z)) = \frac{\partial}{\partial z}F_1 = \frac{1}{z - q^*} = \frac{q^*}{\sigma^2}$$

## B.2 Resolvent Matrix

### B.2.1 General Case

$$M^{-1} = \frac{1}{Z}\int d^N\psi\ \psi\psi^T \exp\left[-\frac{1}{2}\psi^T M\psi\right]$$

$$= \lim_{m\to -1} Z^m \int d^N\psi\ \psi\psi^T \exp\left[-\frac{1}{2}\psi^T M\psi\right]$$

$n = m+1$

$$= \lim_{n\to 0}\int d^{Nn}\vec{\psi}_\alpha\ \psi_1\psi_1^T \exp\left[-\frac{1}{2}\psi_\alpha^T M\psi_\alpha\right] =: \langle \psi_1\psi_1^T\rangle_{n=0}$$

For Gaussian Wigners can use S.P. config

$$\mathbb{E}[G_X(z)]_{ij} = \langle \psi_{1i}\psi_{1j}\rangle = -2\lim_{n\to 0}\frac{\partial}{\partial(z\delta_{ij})}\frac{z^n - 1}{n} = \delta_{ij}\, g_X(z)$$

### B.2.2 Free Addition

$$C = A + OBO^T$$

Want: $\mathbb{E}_O\left[(z\mathbb{I} - A - OBO^T)^{-1}\right]$

WLOG $A, B$ diagonal

$$\mathbb{E}\, G_C(z) = \lim_{n\to 0}\int \frac{d^{Nn}\vec{\psi}_\alpha}{(2\pi)^{Nn/2}}\ \psi_1\psi_1^T \exp\left[-\frac{\psi_\alpha^T(z\mathbb{I}-A)\psi_\alpha}{2}\right]\mathbb{E}_O \exp\left[\frac{1}{2}\psi_\alpha^T OBO^T\psi_\alpha\right]$$

$$\mathbb{E}_O \exp\left[\frac{N}{2}\sum_\alpha \mathrm{Tr}\, Q_{\alpha\alpha} OBO^T\right]$$

low rank ⇒ HCIZ low rank formula applies

$$\delta\left(\frac{\psi\psi^T}{N} - Q\right) = \int_{-i\infty}^{i\infty}\frac{N^{\frac{n(n+1)}{2}}}{2^n}\frac{d\hat{Q}}{(2\pi)^{N/2}}\exp\left[-\frac{N}{2}\mathrm{Tr}\, Q\hat{Q} + \frac{1}{2}\mathrm{Tr}\,\hat{Q}\psi\psi^T\right]$$

?

$$\exp\left[\frac{N}{2}\mathrm{Tr}_n\, H_B(Q)\right]$$

$$\int_0^Q R_B(Q)$$

Can now do $\psi$ integral

totally diagonal over $k$

$$J_{ij} = \int \frac{d^{Nn}\vec{\psi}_\alpha}{(2\pi)^{Nn/2}}\ \psi_{1i}\psi_{1j}\exp\left[-\frac{1}{2}\psi_{\alpha k}(z\delta_{\alpha\beta} - a_k\delta_{\alpha\beta} - \hat{Q}_{\alpha\beta})\psi_{\beta k}\right]$$

$$= -\frac{1}{2}\prod_{k=1}^N \det\left[(z-a_k)\delta_{\alpha\beta} - \hat{Q}_{\alpha\beta}\right] + \left[(z - a_i - \hat{Q})^{-1}\right]_{11}^{n\times n}\delta_{ij}$$

why?

will imply $G_C$ is diagonal

$$\Rightarrow [\mathbb{E}\, G_C]_{ij} = \lim_{n\to 0} \underbrace{\int d[Q,\hat{Q}]}_{\text{consts here don't matter for this calc'n}} \delta_{ij} \underbrace{[(z-a_i-\hat{Q})^{-1}]_{11}}_{\text{no power of } N \Rightarrow \text{ does not determine saddle}} \exp\Big[\underbrace{\frac{N}{2}\Big[\mathrm{Tr}\, H_B(Q) - \mathrm{Tr}\, Q\hat{Q} - \frac{1}{N}\sum_{k=1}^{N} \mathrm{Tr}\log(z-a_k-\hat{Q})\Big]}_{\text{saddle } \frac{N}{2} S_{\text{eff}}}\Big]$$

$$\frac{\partial S_{\text{eff}}}{\partial Q_{\alpha\beta}} = -\hat{Q}_{\alpha\beta} + R_B(Q)_{\alpha\beta} \Rightarrow \boxed{\hat{Q} = R_B(Q)}$$

$$\frac{\partial S_{\text{eff}}}{\partial \hat{Q}_{\alpha\beta}} = -Q_{\alpha\beta} + \frac{1}{N}\sum_{k=1}^{N} \left[(z-a_k)\mathbb{1}_n - \hat{Q}\right]^{-1}_{\alpha\beta}$$

Work in basis that diagonalizes $Q$

$$Q_{\alpha\alpha} = \frac{1}{N}\sum_{k=1}^{N} \frac{1}{z-a_k-\hat{Q}_{\alpha\alpha}} =: g_A(z-\hat{Q}_{\alpha\alpha})$$

at large $z$ the solution is unique $\forall \alpha \Rightarrow Q = q^*\mathbb{1} \quad \hat{Q} = \hat{q}^*\mathbb{1}$

$$\hat{q} = R_B(q) \qquad q = g_A(z-\hat{q})$$

For large $z$ $q, \hat{q}$ are small $\Rightarrow$ justifies deforming $\hat{Q}$ contour How?

$$\Rightarrow \mathbb{E}\, G_C(z) = \lim_{n\to 0} \frac{\delta_{ij}}{z-a_i-\hat{q}^*} \cancel{\exp\Big[\frac{nN}{2}\Big[-q^*\hat{q}^* + H_B(q^*) - \frac{1}{N}\sum_{k=1}^{N}\log(z-a_k-\hat{q}^*)\Big]\Big]}$$

$$\Rightarrow E\, G_C(z) = G_A(z-R_B(q^*)) \qquad \left. q^* = g_A(z-R_B(q^*)) \right\} \ast$$

13.2.3

After taking $\tau$ of $\ast$ we get

$$\underbrace{q^* = g_C(z) = g_A(z-R_B(q^*))}_{\text{subordination}} \qquad q^* = g(z-R_B(q^*))$$

$\Rightarrow \ast$ is a more powerful version!

$$\mathbb{E}\, G_C(z) = G_A(z - R_B(g_C(z)))$$
$$= G_B(z - R_A(g_C(z)))$$

Can use replica theory to show for $C = A^{1/2} B A^{1/2}$

$$\mathbb{E}\, T_C(\xi) = T_A[S_B(t_C(\xi))\xi]$$
$$\Rightarrow \mathbb{E}\, G_C(z) = S^* G_A(z S^*) \qquad S^* = S_B(z g_C(z) - 1)$$

## 13.2.4

Because of strong repulsion $\lambda_i$ barely fluctuate

$\Rightarrow$ Annealed approximation $\langle \log Z \rangle \approx \log \langle Z \rangle$ is exact for getting

$$g_X(z)$$
$$\mathbb{E}G = \langle \psi\psi^T \rangle$$

$$\mathbb{E}\, Z^n = \exp\left(-\frac{nN}{2} F_1\right)$$
$$\Rightarrow \frac{1}{N} \log \mathbb{E} Z = -\frac{1}{2} F_1 \quad \text{as required}$$

This will not be the case for the following rank 1 HCIZ integral beyond some range

## 13.3 Rank 1 HCIZ

$$T = t\hat{e}_1$$

$$H_B(t) := \lim_{N\to\infty} \frac{2}{N} \log \left\langle \exp\left[\frac{N}{2} \mathrm{Tr}\, TOBO^T\right] \right\rangle_O$$

Will then average $H_B$ over $B$

Annealed approx:

$$\hat{H}(t) := \lim_{N\to\infty} \frac{2}{N} \log \left\langle \exp\left[\frac{N}{2} \mathrm{Tr}\, TOBO^T\right] \right\rangle_{O,B}$$

For small $t$ $\quad \hat{H}(t) = H_B(t) = \langle H_B(t) \rangle_B$

Beyond $t_c$ there is a phase transition

### 13.3.1 Annealed Average

$$\left\langle \exp\left[\frac{N}{2} \operatorname{Tr} TX\right] \right\rangle_X = \left\langle \exp\left[\frac{Nt}{2} e_1^T X e_1\right] \right\rangle_X$$

$$= \int \frac{dX_{11}}{\sqrt{4\pi\sigma^2/N}} \exp\left[\frac{Nt}{2} X_{11} - \frac{N}{4\sigma^2} X_{11}^2\right]$$

$$= \exp\left[\frac{N t^2 \sigma^2}{4}\right]$$

$$\Rightarrow \hat{H}_{wig}(t) = \frac{t^2 \sigma^2}{2}$$ $\Leftarrow$ same as $\int R_X(t)\, dt$

### 13.3.2 Quenched Average

Set $\sigma^2 = 1$ for simplicity

$$Z_t(X) = \int \frac{d^N \psi}{(2\pi)^{N/2}} \delta(Nt - |\psi|^2) \exp\left(\frac{1}{2} \psi^T X \psi\right)$$

$\text{Vol } S_{N-1}/(2\pi)^{N/2} = e^{N/2}\, t^{N/2}$

$\sim$ $\qquad \Rightarrow \frac{2}{N} \log \ldots = 1 + \log t$

$$\mathbb{E}\, H_X(t) = \lim_{N \to \infty} \frac{2}{N} \lim_{n \to 0} \frac{Z_t^n(X) - 1}{n} - \frac{2}{N} \log Z_t(0)$$

$$Z_t^n(X) = \int \frac{d^{Nn}\psi_\alpha}{(2\pi)^{Nn/2}} dz_\alpha \exp\left[\frac{1}{2} \psi_\alpha^T X \psi_\alpha + \frac{1}{2}\left(N z_\alpha t - z_\alpha \psi^T \psi\right)\right]$$

$$\mathbb{E}_X \exp\left(\frac{1}{2} \psi^T X \psi\right) = \mathbb{E}_i \exp\left(\frac{1}{2} \overset{1/N}{X_{ii}} \psi_\alpha^i \psi_\alpha^i\right) \mathbb{E}_{i<j} \exp\left(\overset{1/2N}{X_{ij}} \psi_\alpha^i \psi_\alpha^j\right)$$

$$= \exp\left[\frac{1}{4N} \sum_i \left(\sum_\alpha \psi_\alpha^i \psi_\alpha^i\right)^2\right] \exp\left[\frac{1}{2N} \sum_{i<j} \left(\sum_\alpha \psi_\alpha^i \psi_\alpha^j\right)^2\right]$$

$$= \exp\left[\frac{1}{4N} \sum_{i,j} \left(\sum_\alpha \psi_\alpha^i \psi_\alpha^j\right)^2\right]$$

$$= \exp\left[\frac{1}{4N} \sum_{\alpha,\beta} \left(\sum_i \psi_\alpha^i \psi_\beta^i\right)^2\right]$$ $\Leftarrow$ now Hubbard

$$= \cancel{\#}\int dq\, \exp\left[-\frac{N}{4}\mathrm{Tr}\, q^2 + \sum_{\alpha\beta} \frac{q_{\alpha\beta}\,\psi_\alpha^T\psi_\beta}{2}\right]$$

$c(n)$ numerical coeff ← $N$-indep ⇒ no contrib

Do $\psi$ integral

$$\int \frac{d^{Nn}\psi_\alpha}{(2\pi)^{Nn}} \exp\left[-\frac{(z_\alpha\delta_{\alpha\beta} - q_{\alpha\beta})\,\psi_\alpha^T\psi_\beta}{2}\right] \quad \leftarrow \text{decouples over } i$$

$$= \exp\left[-\frac{N}{2}\,\mathrm{Tr}\log \underset{\text{diag } z_\alpha}{z} - q\right]$$

$$\Rightarrow \mathbb{E}[z_1^n] = \cancel{\#}\int dq\,dz\, \exp\left[\frac{N}{2}\underbrace{\left[t\,\mathrm{Tr}\, z - \frac{\mathrm{Tr}\, q^2}{2} - \mathrm{Tr}\log z - q\right]}_{F_n(q,z;t)}\right]$$

$n=1 \Rightarrow \quad tz - \frac{q^2}{2} - \log z - q$

$$\partial_q = 0 \Rightarrow q = \frac{1}{z-q} \quad \Rightarrow \quad q = t \qquad \Rightarrow F_1^*(t) = t^2 + 1 - \frac{t^2}{2} + \log t$$

$$\partial_z = 0 \Rightarrow t = \frac{1}{z-q} \qquad\qquad z = t + t^{-1} \qquad \Rightarrow H_{\text{wig}} = t^2/2 \quad ✓$$

General $n$

$$\partial_q = 0 \Rightarrow q = (z-q)^{-1}$$
$$\partial_z = 0 \Rightarrow t = [(z-q)^{-1}]_{\alpha\alpha} \qquad \Rightarrow q = \begin{pmatrix} t & b & \cdots & b \\ b & t & \cdots & b \\ \vdots & & \ddots & b \\ b & \cdots & & t \end{pmatrix}$$

$$q = (t-b)\mathbb{1} + nb\frac{\mathbf{1}\mathbf{1}^T}{n} \Rightarrow \lambda_q = \begin{cases} t-b+nb & \times 1 \\ t-b & \times(n-1) \end{cases}$$

$$\Rightarrow (z-q)^{-1} = \frac{\mathbb{1}}{z-t+b} + \frac{1}{(z-t+b)^2}\,\frac{b\,\mathbf{1}\mathbf{1}^T}{1 - nb(z-t+b)^{-1}}$$

$$\Rightarrow t-b = \frac{1}{z-t+b} \quad \Rightarrow z = (t-b) + (t-b)^{-1}$$

$$\Rightarrow b = \frac{(t-b)^2 b}{1-nb(t-b)}$$

1) $b=0 \Rightarrow \quad z = t + t^{-1}$

2) $b \neq 0 \Rightarrow \quad (b-t)^2 - nb(b-t) - 1 = 0$

$$b^2(1-n) + (n-2)tb + t^2 - 1 = 0$$

$$\Rightarrow b = \frac{(n-2)t}{2(n-1)} \pm \frac{\sqrt{(n^2 \cancel{-4n+4})t^2 - 4(\cancel{t^2}-1)(1-n)}}{2(n-1)}$$

$$= \frac{(n-2)t \pm \sqrt{n^2t^2 - 4(n-1)}}{2(n-1)}$$

$$z = (t-b) + (t-b)^{-1}$$

$$= \frac{n^2 t \pm (n-2)\sqrt{n^2t^2 - 4(n-1)}}{2(n-1)} =: z_\pm$$

$n=1 \Rightarrow \frac{nt(n \pm (n-2))}{2(n-1)}$ $\begin{cases} z_+ = t \\ z_- \text{ undef} \end{cases}$ $\Rightarrow$ annealed

$n \geq 2 \Rightarrow t \leq \frac{2\sqrt{n-1}}{n}$ has no solutions $\Rightarrow z_0$ is solution there

$t > t_s := \frac{2\sqrt{n-1}}{n}$ has $z_+ > z_-$ always $\Rightarrow$ compare $z_0$ & $z_+$

$$z_0 = t + t^{-1} \quad \text{vs} \quad \frac{n^2 t}{2(n-1)} + \frac{(n-2)}{2(n-1)}\sqrt{n^2t^2 - 4(n-1)}$$

$n=2 \Rightarrow t_s = 1 \qquad z_+ = 2t \Rightarrow t + t^{-1} < 2t$ when $t^{-1} < t \Rightarrow t > 1$

$\Rightarrow t_c = 1$

For $n > 2 \qquad t_s < t_c < 1$

$$\frac{d}{dt} F_n = \frac{\partial F_n}{\partial t} = \operatorname{Tr} z^* = n z^*$$

$\Rightarrow$ For $n \geq 1 \qquad \log \mathbb{E} Z_t^n \sim \frac{N_n}{2} \mathcal{F}_n$

$$\partial_t \mathcal{F}_n = \begin{cases} t + t^{-1} & t < t_c \\ \frac{n^2 t + (n-2)\sqrt{n^2t^2 - 4(n-1)}}{2(n-1)} \end{cases} \Rightarrow \partial_t \mathcal{F}_0 = \begin{cases} t + t^{-1} & t < t_c \\ 2 & t > t_c \end{cases}$$

we must have $t_c = 1$ for $\mathcal{F}_1$ to be continuous

Note $n \to 0$ flips maximizing $\mathcal{F}$ w/ minimizing it!

$$\Rightarrow \frac{d}{dt} \mathop{\mathbb{E}}_{X} H_X(t) = \begin{cases} t & t \le 1 \\ 2 - t^{-1} & t > 1 \end{cases}$$

$$\Rightarrow H_X(t) = \begin{cases} t^2/2 & t \le 1 \\ 2t - \log t & t > 1 \end{cases}$$

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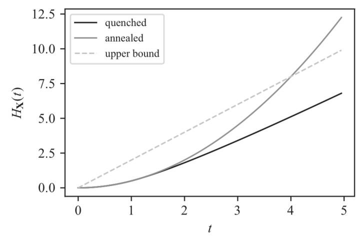



Figure 13.2 The function $H_X(t)$ for a unit Wigner computed with a "quenched" average (HCIZ integral) and an "annealed" average. We also show the upper bound given by Eq. (10.59). The annealed and quenched averages are identical up to $t = t_c = 1$ and differ for larger $t$. The annealed average violates the bound, which is expected as in this case $\lambda_{max}$ fluctuates and exceptionally large values of $\lambda_{max}$ dominate the average exponential HCIZ integral.

## 13.4 Spin Glasses & Low-Rank HCIZ

$$\mathcal{H}(\{S\}) = \frac{1}{2} \sum_{i,j=1}^{N} J_{ij} S_i S_j \qquad J = O \Lambda O^T$$

Take $J = X$ Wigner $\Rightarrow$ SK model

$$F := -T \,\mathbb{E}_J \log Z \qquad Z = \mathop{\mathrm{Tr}}_{S} e^{\mathcal{H}(\{S\})/T}$$

$$\mathop{\mathbb{E}}_{J} Z^n = \mathop{\mathbb{E}}_{J} \sum_{S^1 \dots S^n} \exp\left[\frac{1}{2T} \sum_{\alpha, i, j} J_{ij} S_i^\alpha S_j^\alpha\right]$$

$K_{ij}^{(n)} := \sum_\alpha \frac{S_i^\alpha S_j^\alpha}{N}$ is a rank $n \ll N$ matrix

$\Rightarrow$ HCIZ yields

$$\mathop{\mathbb{E}}_{J} \exp\left[\frac{N}{2T} \mathrm{Tr}\, K^{(n)} \underbrace{O \Lambda O^T}_{J_{ij}}\right] \approx \exp\left[\frac{N}{2} \mathrm{Tr}_N\, H_J(K^n/T)\right]$$

$$= \exp\left[\frac{N}{2} \mathrm{Tr}_n\, H_J(Q/T)\right]$$

$$\Rightarrow \mathbb{E}\, Z^n = \sum_{S^1 \dots S^n} \exp\left[\frac{N}{2} \mathrm{Tr}_n\, H_J(Q/T)\right]$$

$$= \sum_{S^1 \dots S^n} * \int dQ\, d\hat{Q} \exp\left[\frac{N}{2} \mathrm{Tr}\, H_J\left(\frac{Q}{T}\right) - N \,\mathrm{Tr}\, Q\hat{Q} + \sum_{\alpha\beta} \hat{Q}_{\alpha\beta} (S^\alpha)^T S^\beta\right]$$

decouples over $i$

$$= \# \int d[Q,\hat{Q}] \exp\left[\frac{N}{2} \operatorname{Tr} H_J\left(\frac{Q}{T}\right) - N \operatorname{Tr} Q\hat{Q} + N G(\hat{Q})\right]$$

$$G(\hat{Q}) = \log \sum_{\underset{\text{scalar}}{s^1 \dots s^n}} e^{s^T \hat{Q} s}$$

Saddle:

$$\partial_Q \Rightarrow \hat{Q} = \frac{1}{2T} R_J(Q/T)$$

SCEq: $$\partial_{\hat{Q}} \Rightarrow Q_{\alpha\beta} = \langle s^\alpha s^\beta \rangle_T := \frac{1}{Z} \sum_s s^\alpha s^\beta \exp\left[\frac{1}{2T} s^T R_J\left(\frac{Q}{T}\right) s\right]$$

Replica Symmetry:

$$Q_{\alpha\beta} = (1-q)\mathbb{1} + q \mathbf{1}\mathbf{1}^T \Rightarrow \lambda_Q = \begin{cases} 1+(n-1)q & \times 1 \\ 1-q & \times(n-1) \end{cases}$$

$$\Rightarrow R_J(Q) = \begin{cases} R_J\left(\frac{1+(n-1)q}{T}\right) \\ R_J\left(\frac{1-q}{T}\right) \end{cases} \Rightarrow \begin{aligned} r &= n^{-1}\left(R_B\left[\tfrac{1+(n-1)q}{T}\right] - R_B\left[\tfrac{1-q}{T}\right]\right) \\ r_d &= R_J\left[\tfrac{1-q}{T}\right] + r \end{aligned}$$

$$Z = \sum_s \exp\left[\frac{1}{2T}\left(n r_d + r \underbrace{\sum_{\alpha\neq\beta} s^\alpha s^\beta}_{\left(\sum_\alpha s^\alpha\right)^2 - n}\right)\right] = \sum_s e^{\frac{n}{2T}(r_d - r)} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2} + x\sqrt{\frac{r}{T}}\sum_\alpha s_\alpha}$$

$$= e^{\frac{n}{2T}(r_d - r)} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} \left[2\cosh x\sqrt{\tfrac{r}{T}}\right]^n$$

$$\Rightarrow \log Z = \frac{n}{2T}(r_d - r) + n\log 2 + \log \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} \cosh^n\left(x\sqrt{\tfrac{r}{T}}\right)$$

$$2T \frac{\partial \log Z}{\partial r} = \sum_{\alpha\neq\beta} \langle s_\alpha s_\beta \rangle = n(n-1)q$$

$$= -n + \frac{n}{Z} \frac{2T}{2\sqrt{rT}} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} x \frac{\sinh\left(\sqrt{\frac{r}{T}}x\right)}{\cosh\left(\sqrt{\frac{r}{T}}x\right)^{1-n}}$$

$$= -n + \frac{n}{\underset{n\to 0}{Z}} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{\cosh\left(\sqrt{\frac{r}{T}}x\right)^2} = -n \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} \tanh^2\left(\sqrt{\tfrac{r}{T}}x\right)$$

$$\Rightarrow q = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} \tanh\left(\sqrt{\tfrac{r}{T}}x\right) \quad \Big\} \; \left\langle \tanh^2 \sqrt{\tfrac{r}{T}} x \right\rangle_{x\sim N(0,1)}$$

From before: $r = n^{-1}\left( R_B\left[\frac{1+(n-1)q}{T}\right] - R_B\left[\frac{1-q}{T}\right]\right)$

$$\lim_{n\to 0} r = \frac{q}{T} R_B'\left[\frac{1-q}{T}\right]$$

$$\underbrace{q=0 \Rightarrow r=0}_{\text{uncorrelated phase (high } t)} \Rightarrow q=0 \quad \in SC$$

$q \neq 0$ but small ← assumes that phase transition is continuous

$$r \approx \frac{q}{T} R'\left(\frac{1}{T}\right) - \frac{q^2}{T^2} R''\left(\frac{1}{T}\right)$$

$$\Rightarrow \left\langle \tanh^2 \sqrt{\tfrac{r}{T}}\, x \right\rangle \approx \overset{1}{\cancel{\langle x^2\rangle}} \frac{r}{T} - \frac{2}{3} \overset{3}{\cancel{\langle x^4\rangle}} \frac{r^2}{T^2}$$

$$\approx \frac{q}{T^2} R_J'\left(\frac{1}{T}\right) \underbrace{- \frac{q^2}{T^3} R_J''\left(\frac{1}{T}\right) - \frac{2q^2}{T^4} R'\left(\frac{1}{T}\right)^2}_{\text{assume } <0}$$

(true for $R_J(x)=x$ in Wigner case)

$$q \approx \frac{q}{T^2} R_J'\left(\frac{1}{T}\right) + (>0)$$

$$T_c^2 = R_J'\left(\frac{1}{T_c}\right) = \frac{1}{T_c} \quad \text{for Wigner}$$

$$\Rightarrow T_c = 1$$

For random orthogonal model where $\lambda_J = \pm 1$ equally
transition is discontinuous

Below $T_c$ we have stages of RSB

# Chapter 14: Edge Eigenvalues & Outliers

## 14.1 Tracy-Widom

$\lambda_{max}$ maximum eig of Wigner or Wishart

$\lambda_+$ denotes upper edge

* $P[\lambda_{max} \leq \lambda_+ + \gamma N^{-2/3}] = F_1(u)$

$\beta=1$ Tracy-Widom

Wigner: $\lambda_+ = 2 \quad \gamma = 1$

Wishart: $\lambda_+ = (1+\sqrt{q})^2 \quad \gamma = \sqrt{q}\, \lambda_+^{2/3}$

* Holds for symmetric IID matrices w/ finite 4th moment

$f_1 := F_1' \rightsquigarrow \log f_1 \propto -u^{3/2} \quad u \to \infty$

$\propto -|u|^3 \quad u \to -\infty$ ← much thinner (harder to push $\lambda_{max}$ in & compress the spectrum)

From 5.4.2: $\sqrt{V'(s)^2 - 4\Pi(s)} \approx C(s-\lambda_+)^\theta$

$$\Rightarrow \Phi(x) = \int_{\lambda_+}^x \sqrt{V'(s)^2 - 4\Pi(s)} \approx \frac{C}{\theta+1}(s-\lambda_+)^{\theta+1}$$

$$\Rightarrow P(\lambda_{max}) = \exp\left[-\frac{N\beta}{2}\Phi(\lambda_{max})\right] = \exp\left[-\frac{2\beta C}{3} u^{3/2}\right] \qquad u = N^{2/3}(\lambda_{max} - \lambda_+)$$

I think they mean $1/3$

If $\rho(\lambda) \to 0$ as $(\lambda_+ - \lambda)^\theta$

The probability $P(\lambda > \lambda_*) \propto (\lambda_+ - \lambda_*)^{\theta+1}$

When this is $< 1/N \Rightarrow |\lambda_+ - \lambda_*| \sim N^{-\frac{1}{1+\theta}}$

For Gaussian ensembles

$$\rho_N(\lambda \approx \lambda_+) = N^{-1/3} \Phi_1(N^{2/3}(\lambda - \lambda_+))$$

as $u \to -\infty \quad \Phi_1 \sim |u|^{-1/2}$

## 14.2 Additive Low-Rank

$M \Rightarrow M + a v v^T \qquad |v| = 1$

$G_a = (z - M - a u u^T)^{-1} = G(z) + a \dfrac{G u u^T G}{1 - a u^T G u}$

$G_a$ has a pole $\Rightarrow 1 - a u^T G(\lambda_1) u = 0 \qquad G, u u^T$ free

$u^T G(z) u = Tr[G(z) u u^T] = \dfrac{Tr[G]}{N} = g_N(z) \rightarrow g(z)$

pole when $g(\lambda_1) = 1/a$

$\Rightarrow \lambda_1 = \mathfrak{z}(1/a)$

monotonically increasing in a

$\lambda_1 = \lambda_+$ when $a = g(\lambda_+)^{-1}$

critical value of a

Generally $\dfrac{d\mathfrak{z}(g_+)}{dg} = 0$ ie for Wigner $\mathfrak{z}(g) = \sigma^2 g + g^{-1}$

$\Rightarrow \dfrac{\partial}{\partial g} \mathfrak{z}(g) = \sigma^2 - g^{-2} \Rightarrow g = \sigma^{-1}$

$\mathfrak{z}(\sigma^{-1}) = 2\sigma = \lambda_+$

$\lambda_1 = a + R(1/a)$

$\Rightarrow \lambda_1 = a + \tau(M) + \dfrac{\kappa_2(M)}{a} + \dots$

For Wigner $\lambda_1 = a + \sigma^2/a \qquad a > a^* = \sigma$

$\left.\dfrac{d\lambda_1}{da}\right|_{a^*} = \left.\dfrac{d\mathfrak{z}}{dg}\right|_{g_+ = 1/a^*} = 0$

$\Rightarrow \lambda_1 = \lambda_+ + (a - a^*)^2 + \dots$ For square root singularities

Fluctuations of the outlier around $\lambda_1$ can be shown to go as $N^{-1/2}$

While fluctuations of $\lambda_{max} \sim N^{-2/3}$

Transition is called BBP transition

1)

2) For rank k, can show

$$\lambda_k = a_k + R\left[\frac{1}{a_k}\right] \qquad a_k > 1/g_+$$

### 14.2.2 Outlier Eigenvectors

For $a \gg 1$ $\quad v \approx u$

For $a < 1$ $\quad u$ will mix w/ the rest of M's evecs

$$G_a = \sum \frac{v_i v_i^T}{z - \lambda_i} \Rightarrow \lim_{z \to \lambda_1} u^T G_a u \, (z - \lambda_1) = |v_1^T u|^2$$

$$= \lim_{z \to \lambda_1} \left[ g(z) + a \frac{g(z)^2}{1 - a g(z)} \right] (z - \lambda_1)$$

$$= \lim_{z \to \lambda_1} \frac{g(z)(z - \lambda)}{1 - a g(z)} = \lim_{z \to \lambda_1} \left[ -\frac{g(z)}{a g'(z)} \right] \qquad \frac{1}{a} = g(z)$$

$$\Rightarrow |v_1^T u|^2 = -\frac{g(z)^2}{g'(z)} \quad *$$

$$z = R(g(z)) + g^{-1}(z)$$

$$\Rightarrow 1 = R'(g(z)) g'(z) - g^{-2}(z) g'(z) \Rightarrow \frac{1}{g'(z)} = R'(g) - \frac{1}{g^2}$$

$$\Rightarrow |v_1^T u|^2 = 1 - g^2 R'(g)$$

$$= 1 - \frac{R'\left[\frac{1}{a}\right]}{a^2}$$

$$a \to \infty \Rightarrow R'[0] \sim \kappa_2(M)$$

$$\Rightarrow |v^T u|^2 = 1 - \frac{\kappa_2(M)}{a^2}$$

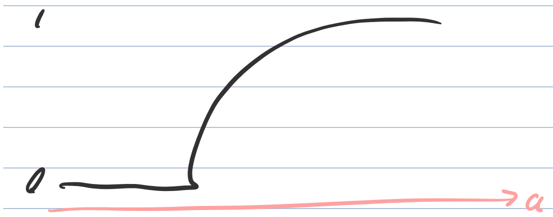
As $\lambda_1 \to \lambda_+$ $\qquad g(z) = \int_{\lambda_-}^{\lambda_+} \frac{p(x)\,dx}{z - x} \Rightarrow g'(z) = -\int_{\lambda_-}^{\lambda_+} \frac{p(x)\,dx}{(z-x)^2}$

For $p(\lambda) \sim (\lambda_+ - \lambda)^\theta$ $\quad \theta < 1$ $\quad g(\lambda_+)$ Finite
but $g'$ diverges as $(z - \lambda_1)^{\theta - 1}$

$* \Rightarrow |v_1^T u|^2 \propto (\lambda_1 - \lambda_+)^{1-\theta} \qquad$ as $a_* \to a$

For Wigner $R(x) = \sigma^2 x \Rightarrow R' = \sigma^2$

$$\Rightarrow |v_1^T u|^2 = 1 - \underbrace{g^2}_{zg-1}\sigma^2 = 2 - \frac{z^2 - \sqrt{1 - 4\sigma^2/z^2}}{2\sigma^2} = 1 - \left(\frac{a^*}{a}\right)^2 \qquad a > a^* = \sigma$$



The fact that $|u^T v|^2 \sim O(1)$ for $\lambda_1 > \lambda_+$ is not general (see Ch 19)

## Ex 14.2.1

a) $\frac{\mathbb{1}\mathbb{1}^T}{N}$

b) $\lambda_1 = a + \underbrace{R_W\left[\frac{1}{a}\right]}_{(1-qa^{-1})^{-1}} = a + \frac{1}{1-qa^{-1}} = \frac{1+a-q}{1-qa^{-1}}$

c) $|v^T u|^2 = 1 - a^{-2} R'(a^{-1})$

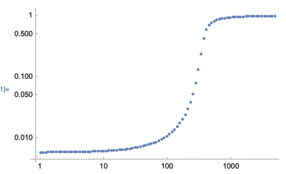
$$R' = (1-qg)^{-2} q$$

$$= 1 - \frac{q}{(a-q)^2}$$

d)

```
In[1265]:= n = 100;
W = RandomVariate[WishartMatrixDistribution[4 n, IdentityMatrix[n]]];

One = ConstantArray[1, {n, n}] / n;
uv2 = {};
as = {};
For[a = 1, a ≤ 5000, a = a * 1.1,
 M1 = W + a One;
 as = as ~ Join ~ {a};
 uv2 = uv2 ~ Join ~ {(ConstantArray[1 / Sqrt[n], {n}].Eigensystem[M1][[2]][[1]])^2};]
ListPlot[Transpose[{as, uv2}], ScalingFunctions → {"Log", "Log"}]
```

Out[1271]=



## 14.3 Fat Tails

Let $X_{ij}$ come from wide-tailed dist

$X_{ij} = \frac{x_{ij}}{\sqrt{N}}$ $\quad x_{ij} \sim P(x)$ of Mean $=0$, Var $=1$ $\quad$ decays as $\mu |x|^{-1-\mu}$

$$P(|X_{ij}|>1) \approx 2\int_{\sqrt{N}}^{\infty} dx \, \frac{\mu}{x^{1+\mu}} = \frac{2}{N^{\mu/2}}$$

$\Rightarrow$ # entries $>1$ $\approx \frac{N(N+1)}{N^{\mu/2}} \sim N^{2-\mu/2}$

$\mu > 4 \Rightarrow 0$
$\Rightarrow$ No outliers

Finiteness of $x_4$ is enough for Tracy-Widom!

Even for $\mu > 4$ large elements still dominate tail at finite $N$
for such power law distributions

When $2 < \mu < 4$ $\quad x_2$ is finite

$\Rightarrow \rho(\lambda) \Rightarrow$ semicircle by CLT

$\mathcal{O}(N^{2-\mu/2})$ outliers with density $\quad X_{ij} = X_{ji} = a \gg 1$ contributes $\lambda = \pm(a + a^{-1})$

$$\rho_{out}(\lambda > 2) = N^{1-\mu/2} \int_1^{\infty} dx \, \frac{\mu}{x^{1+\mu}} \, \delta(\lambda - x - \tfrac{1}{x})$$

$\rho \to 0$ as $N \to \infty$
$\lambda_{max} \to \infty$

$\mu < 2 \Rightarrow$ "Levy"

$X_{ij} = \frac{x_{ij}}{N^{1/\mu}}$ $\quad P(x) \sim \frac{\Gamma(1+\mu)}{|x|^{1+\mu}} \sin \frac{\pi\mu}{2}$ $\quad$ as $x \to \infty$

$\rho(\lambda)$ becomes unbounded at $N \to \infty$

Tail of $\rho(x)$ turns out to be the same
as the tail of $P(x)$

## 14.4 Multiplicative Perturbation

True → Sample cov is free prod w/ white Wishart

$$E = B^{1/2} C_0^{1/2} (1 + \alpha u u^T) C_0^{1/2} B^{1/2}$$

Wish    true    evec of $C_0$

$\tau(B) = 1$    $B$ is a white wishart ⇒ Noisy obs of pert. cov w/ one mode amplified by $1+\alpha$

$$E_0 = B^{1/2} C_0 B^{1/2}$$

$$\det(\lambda_1 \mathbb{1} - E_0 - a B^{1/2} C_0 u u^T B^{1/2}) = 0$$

$\lambda_1$ outside spec $E_0$ ⇒ invert $\lambda_1 \mathbb{1} - E_0$

$$\Rightarrow \det(\cdots) = \det(\lambda_1 \mathbb{1} - E_0)\left(1 - a\, \underbrace{u^T C_0^{1/2}}_{\lambda_0 u^T} B^{1/2} G_0(\lambda_1) B^{1/2} C_0^{1/2} u\right) = 0$$

free

$$\Rightarrow a \lambda_0\, u^T B^{1/2} G_0 B^{1/2} u = 1$$

$$\Rightarrow a \lambda_0\, N^{-1} \operatorname{Tr}[G_0 B] = 1$$

Using $\mathbb{E}\, G_C = S^* G_A(z S^*)$    $S^* = S_B(z g_C - 1)$    for $C = B^{1/2} A B^{1/2}$

For $E_0 = B^{1/2} C_0 B^{1/2} \Rightarrow \mathbb{E}\, G_0 = S^* G_B(z S^*)$    $S^* = S_{C_0}(z g_0 - 1)$

$$\Rightarrow \tau[G_0 B] = S^* \tau[G_B(zS^*) B]$$

$$= S^* t_B(\lambda, S^*)$$

$$\Rightarrow a \lambda_0 S^* t_B(\lambda, S^*) = 1$$

Assume $C_0 = \mathbb{1}$ ⇒ $g_0 = \frac{1}{z-1}$ ⇒ $z g_0 - 1 = \frac{1}{z-1} = g_0$    $S^*[g_0] = 1 \;\forall g$

⇒ $\lambda_0 = 1$

$\Rightarrow a\, t_B(\lambda_1) = 1$    → need $t_B$ invertible

$$t_B(\zeta) = \int_{\lambda_-}^{\lambda_+} \frac{\rho_B(x)\, x}{\zeta - x}\, dx$$ $\leftarrow$ decreasing for $\zeta > \lambda_+$

$\lambda_1 = \zeta(a^{-1}) = \dfrac{a+1}{S(a^{-1})}$ $\qquad a > \dfrac{1}{t_B(\lambda_+)}$

Let $B = W$ a Wishart

$S_W = (1+qx)^{-1} \qquad \lambda_\pm = (1 \pm \sqrt{q})^2$

$\Rightarrow \lambda_1 = (a+1)(1+q/a)$
$\approx 1+a+q$

$\Rightarrow a+1$ in the Cov will appear shifted by $q$

## Ex 14.4.1

a) Known

b) $(1+(\sqrt{a}-1)\,uu^T)^2 = 1 + [(\sqrt{a}-1)^2 + 2\sqrt{a} - 2]\, uu^T$

$= 1 + (a-1)\, uu^T$

c) Apply prev result : Spec $B^{1/2}(1+(a-1)uu^T)B^{1/2}$ = Spec $(1+\underbrace{(\sqrt{a}-1)}_{c}uu^T)B(1+\underbrace{(\sqrt{a}-1)}_{c}uu^T)$

$\Rightarrow \lambda_1 = 1 + q + a - 1 = q + a$

$c = \sqrt{a} - 1$
$(c+1)^2 = a$

$q=1 \Rightarrow = \underbrace{1+a}_{\lambda}$

## Ex 14.4.2 Inverse Wishart Mult Pert

$M_p := (1-q)\, W_q^{-1} \qquad p = \dfrac{q}{1-q}$ $\leftarrow$ var of $W^{-1}$

$\Rightarrow S_{M_p}(t) = (1 - pt)$

a) $D = \mathbb{1} + (d-1)\, \hat{e}_1 \hat{e}_1^T$

b) $S_{M_p}(a^{-1}) = (1 - p/a) \Rightarrow \lambda_1 = \dfrac{a+1}{1-p/a} \approx 1 + a + p$

use $d-1 \Rightarrow M_p^{1/2}(\mathbb{1} + (d^2-1)\, uu^T)\, M_p^{1/2}$

$\Rightarrow \alpha = d^2 - 1$

$\lambda_1 \approx d^2 + p$

c)
d) Finish

## 14.5 Phase Retrieval & Outliers

$$\hat{x} = \underset{x}{\operatorname{argmin}} \| (a_k^T x)^2 - y \|_1 \qquad x \in \mathbb{R}^n \qquad k = 1 \dots T$$

Non convex in $x$ w/ many local minima

Need $x_0$ st. $x^* \cdot x_0 \gg 0$

In high dim $\log \mathbb{P}[x^* \cdot x_0 > \varepsilon] \propto -N\varepsilon$

Want nonzero overlap w/ $x^*$

Take $M = \frac{1}{T} \sum_k F(y_k)\, a_k a_k^T$

Assume $a_k \sim N(0, \mathbb{I})$

$\Rightarrow$ WLOG $x^* = \hat{e}_1$

$\Rightarrow f(y_k)$ related to $|a_k \cdot \hat{e}_1|^2$

$$M = \begin{pmatrix} \underbrace{M_{11}}_{1\times 1} & M_{21} \\ M_{12} & M_{22} \end{pmatrix}$$

$$N g_M(z) = \operatorname{Tr} G_{22} + \frac{\operatorname{Tr} G_{22} M_{21} M_{12} G_{22}}{z - M_{11} - M_{12} G_{22} M_{21}}$$

outlier

$$\lambda_1 = M_{11} + \underbrace{M_{12} G_{22}(\lambda_1) M_{21}}_{\text{self-averaging}}$$

$$|v^T x|^2 = \lim_{z\to\lambda_1} \frac{z-\lambda_1}{z-M_{11}-M_{12}G_{22}M_{21}}$$

$$M_{11} = \frac{1}{T}\sum_{k=1}^{T} \mathcal{F}(y_k)(a_k)_1^2 \xrightarrow{T\to\infty} \mathbb{E}[\mathcal{F}(y)(a_1)^2]$$

$$M_{12}G_{22}M_{21} = \frac{1}{T^2}\sum_{k,\ell} \mathcal{F}(y_k)\mathcal{F}(y_\ell)[a_k]_1[a_\ell]_1 \sum_{ij\neq 1} \underbrace{[a_k]_i [a_\ell]_j}_{H} [G_{22}]_{ij}$$

$$\to \frac{1}{T}\sum_{k,\ell} \mathcal{F}_k\mathcal{F}_\ell \underbrace{[a_k]_1[a_\ell]_1}_{\delta_{k\ell}|\vec{a}_1|^2} \left(\frac{HG_{22}H^T}{T}\right)_{k\ell}$$

$$\to q\,\mathbb{E}_a[\mathcal{F}^2[a_1]^2]\,\tau(\underbrace{HH^TG_{22}}_{h(z)}) \qquad q = N/T$$

$$\frac{z-\lambda}{z-c_1-qc_2h(z)}$$

$$\lambda_1 = c_1 + qc_2 h(0)$$

$$|x^*\cdot v_1|^2 = \frac{1}{1-qc_2h'(\lambda_1)}$$

$q\to 0 \Rightarrow T\to\infty$ really fast

$$\Rightarrow M \propto \mathbb{E}\,\mathcal{F}(y)\,\mathbb{1} =: m_1 \qquad \Rightarrow g_M(z) = \frac{1}{z-m_1} \qquad HH^T \to T\cdot\mathbb{1}$$

$$\Rightarrow h(z) = \frac{1}{z-m_1}$$

Linear correction in $q$:

$$\lambda_1 = c_1 + \frac{q\,c_2}{c_1-m_1}$$

why?

FINISH

# Chapter 15 $\boxplus$ and $\boxtimes$: Recipies & Examples

$$g_A(z) = \tau[(z-A)^{-1}]$$

$$\psi_A(z) = \tau[A(\zeta-A)^{-1}] = \tau[(1-\zeta^{-1}A)^{-1}] - 1 = \zeta g_A(\zeta) - 1$$

$$R_A(g) = \mathfrak{z}_A(g) - 1/g \qquad S_A(t) = \frac{t+1}{t\mathfrak{z}_A(t)} \leftarrow \tau(A) \neq 0$$

$$R_{\alpha A}(x) = \alpha R_A(\alpha x) \qquad S_{\alpha A}(t) = \alpha^{-1} S_A(t)$$

$$R_{A+\alpha\mathbb{1}}(x) = \alpha + R_\alpha(x) \qquad S_{A^{-1}}(x) = \frac{1}{S_{A^{-1}}(-x-1)}$$

$$S_A(t) = \frac{1}{R_A(tS_A(t))} \qquad R_A(x) = \frac{1}{S_A(xR_A(x))}$$

$$g_{\mathbb{1}} = \frac{1}{z-1} \Rightarrow \begin{array}{l} R_1(x) = 1 \\ S_1(t) = 1 \end{array}$$

$$\psi_{\mathbb{1}} = \frac{1}{\zeta-1}$$

$$R_A(x) = \sum_{i=1}^{\infty} \kappa_i x^{i-1}$$

$$S_A(x) = \frac{1}{\kappa_1} - \frac{\kappa_2}{\kappa_1^3}x + \frac{2\kappa_2^2 - \kappa_1\kappa_3}{\kappa_1^5}x^2 + \cdots$$

$$\tau(A^{-1}) = [S_{A^{-1}}(0)]^{-1} = S_A(-1)$$

$$\tau(A^{-2}) = -S_{A^{-1}}'(0)[S_{A^{-1}}(0)^3]^{-1} + \tau(A^{-1})^2 = -S_A(-1)^3 \frac{d}{dt}\Big|_{t=0} [S_A(-t-1)]^{-1} + S_A(-1)^2$$

$$= S_A(-1)(S_A(-1) - S_A'(-1))$$

## 15.1.2 Computing $\rho$

$$\boxplus: \quad \begin{array}{l} A \to g_A \to \mathfrak{z}_A \to R_A \searrow \\ \qquad\qquad\qquad\qquad\qquad + R_C \to \mathfrak{z}_C \to g_C \to \rho_C \\ B \to g_B \to \mathfrak{z}_B \to R_B \nearrow \end{array}$$

$C = A+B$

$\eta \to 0\ \mathrm{Im}\ \frac{g(\lambda - i\eta)}{\pi}$

$$x: \quad \begin{matrix} A \to t_A \to \zeta_A \to S_A \searrow \\ \qquad\qquad\qquad\qquad \times S_C \to \zeta_C \to t_C \to \rho_C \\ B \to t_B \to \zeta_B \to S_B \nearrow \end{matrix}$$

$C = \sqrt{A}B\sqrt{A}$

$\eta \to 0^+ \operatorname{Im} \frac{t(x-i\eta)}{\pi}$

can be hard
usually a higher
order poly

Trick: $\pi\rho = \underset{\text{sol'ns}}{\max} \operatorname{Im} \cdots$ for cubic/quadratic since sol'ns come in pairs

## 15.2 R, S Transforms & Useful Ensembles

### 15.2.1 Wigner

$R(x) = \sigma^2 x$

Free sum has $\sigma_c^2 = \sigma_1^2 + \sigma_2^2$

$\tau(X) = 0 \Rightarrow$ No S transform

But $X + m\mathbb{1}$ has $R_{X+m\mathbb{1}} = m + \sigma^2 x$

$\Rightarrow S(t) = \dfrac{1}{m + \sigma^2 t S} \Rightarrow S^2 t\sigma^2 + mS - 1 = 0$

$\Rightarrow S = \dfrac{\sqrt{m^2 + 4t\sigma^2} - m}{2\sigma^2 t} = \dfrac{m}{2\sigma^2 t}\left(\sqrt{1 + \frac{4t\sigma^2}{m^2}} - 1\right)$

$\tau(X^{2k}) = \dfrac{(2k)!}{k!(k+1)!}\sigma^{2k}$ $\to (\sigma\sqrt{t})^{-1}$ as $m \to 0$

$\tau(X^{-k}) = \infty$

### 15.2.2 Wishart

$q = N/T$ $\quad R_{W_q} = \dfrac{1}{1-qx} \Rightarrow \tau(W_q) = 1 \quad \tau(W_q^2) = 1+q$

$g$ satisfies

$\underbrace{g^{-1}}_{\left(\frac{t+1}{z}\right)^{-1}} = z - 1 + q - \underbrace{qzg}_{-qt} \Rightarrow 1 = \left(\frac{t+1}{\zeta}\right)(\zeta - 1) - qt\left(\frac{t+1}{\zeta}\right)$

$\Rightarrow \zeta = (t+1)(\zeta - 1) - qt(t+1)$

$$(t+1)(\zeta-1-qt)-\zeta=0$$

$$\Rightarrow \zeta t-(t+1)(1+qt)=0$$

$$\frac{1}{1-q}+\frac{1}{(1-q)^2}$$

$$\Rightarrow S=\frac{1}{1+qt} \quad \Rightarrow \quad \tau(W_q^{-1})=\frac{1}{1-q} \qquad \tau(W_q^{-2})=\frac{1}{(1-q)^3}$$

## 15.2.3 Inverse Wishart

$$S(-t-1)^{-1} \Rightarrow 1-q-qt$$

$$\tau(W_q^{-1})=\frac{1}{1-q} \quad \Rightarrow \quad M_p \coloneqq (1-q)W_q^{-1}$$

$$S_{M_p}(t)=(1-q)^{-1}(1-q-qt)=1-pt \qquad p=\frac{q}{1-q}$$

$$\Rightarrow \kappa_1(M_p)=S(0)=1$$

$$\kappa_2(M_p^2)=-S'(0)\,S(0)^{-3}=p$$

$$S''(0)=0 \Rightarrow \kappa_3=2\kappa_2^2=2p^2$$

$$R_{M_p}(x)=[S_{M_p}(xR_{M_p})]^{-1}$$

$$\Rightarrow R(1-pxR)=1 \Rightarrow R=\frac{1-\sqrt{1-4px}}{2px}$$

$$\zeta=\frac{t+1}{t(1-pt)} \Rightarrow t+1=\zeta(t-pt^2)$$

$$\Rightarrow p\zeta t^2+(1-\zeta)t+1=0$$

$$\Rightarrow t=\frac{\zeta-1\pm\sqrt{(\zeta-1)^2-4p\zeta}}{2p\zeta}$$

$$g=\frac{t+1}{z}=\frac{(1+2p)z-1-\sqrt{(z-1)^2-4pz}}{2pz^2}$$

$$z\to 0 \Rightarrow \sqrt{1-(2+4p)z+\cdots}=1-(1+2p)z$$

$$\Rightarrow \rho=\frac{\sqrt{(\lambda_+-\lambda)(\lambda-\lambda_-)}}{2\pi p\lambda^2} \qquad \lambda_\pm=2p+1\pm2\sqrt{p(1+p)}$$

$$\tau(M_p^{-1})=\lim_{z\to 0} g_{M_p}(z)=1+p$$

From $z=\infty$ expansion

$$g(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{1+p}{z^3} + \cdots$$

$\tau(M_p)=1$ $\quad$ $\tau(M_p^2)=1+p$

From $z=0$

$$g(z) = -(1+p) - (1+p)(1+2p)z + \cdots$$

$\tau(M_p^{-2}) = (1+p)(1+2p)$

Using $\quad \mathrm{Re}\, g(x) = \fint \frac{\rho(\lambda)}{z-\lambda} d\lambda = \frac{V'(x)}{2}$

$$\Rightarrow V' = \frac{(1+2p)z - 1}{2pz^2} \Rightarrow V_{M_p} = \frac{1}{px} + \frac{1+2p}{p} \log x$$

Recall for Wishart:

$$P(E) = \frac{(T/2)^{NT/2}}{\Gamma_N(T/2)} \frac{(\det E)^{(T-N-1)/2}}{(\det C)^{T/2}} \exp\left[-\frac{T}{2} \mathrm{Tr}[EC^{-1}]\right]$$

cov

$N \times N$ Wishart w/ $T$ datapoints

$$\mathbb{E}[E^{-1}] = \frac{T}{T-N-1} C^{-1} =: \Sigma$$

$T-N-1 \approx \frac{1}{1-q}$

$\mathbb{E}[M] = \Sigma$ $\qquad\qquad$ $M = E^{-1} \Rightarrow \mathrm{Jac} = \det E^{-N-1} = \det M^{N+1}$

$$P(M) = \frac{((T-N-1)/2)^{NT/2}}{\Gamma_N(T/2)} \frac{(\det \Sigma)^{T/2}}{(\det M)^{(T+N+1)/2}} \exp\left[-\frac{T-N-1}{2} \mathrm{Tr}[M^{-1}\Sigma]\right]$$

Jac

$N=1$
$b := \frac{T-2}{2}\Sigma$ $\qquad \Rightarrow P(m) = \frac{b^a}{\Gamma(a)} m^{-a-1} e^{-b/m}$
$a = T/2$

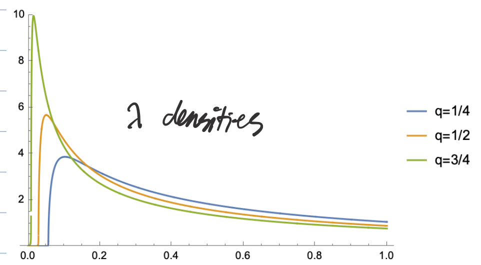
Ex 15.2.1

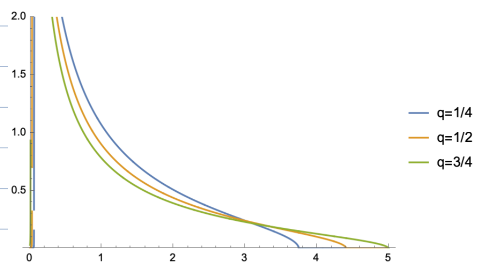
$$E = W_{q_0}^{1/2} W_q W_{q_0}^{1/2}$$ $\qquad W_{q_0}$ is true covar

1) $S_E = S_{W_0} S_W = \dfrac{1}{(1+q_0 t)(1+qt)}$

2) $\Rightarrow \zeta_E = \dfrac{t+1}{t}(1+q_0 t)(1+qt)$ ← cubic to solve for $\zeta$

3) $q_0 \to 0 \Rightarrow$ MP

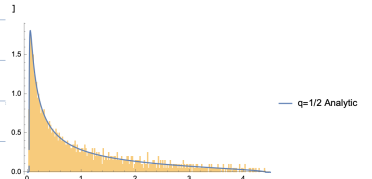



edge of support

4)

```
W0 = RandomVariate[WishartMatrixDistribution[2000, IdentityMatrix[1000] / 2000]];
W1 = RandomVariate[WishartMatrixDistribution[2000, IdentityMatrix[1000] / 2000]];
evals0 = Eigenvalues[W1.W0];
Show[Histogram[evals0, 200, "PDF"],
 Plot[Max[Im@{soln[[1]], soln[[2]], soln[[3]]} /. q → 1 / 2] / π /. z → x - I 0.0001 // Evaluate,
  {x, 0, 10}, PlotRange → {0, 10}, PlotPoints → 1000, PlotLegends → {"q=1/2 Analytic"}]
]
```




# 15.3 Worked-out Examples: Addition

## 15.3.1 Arcsine Law

$M_1, M_2$ symm orth $\Rightarrow \lambda_i = \pm 1$

char poly of $M_1 = (z-1)^{N/2}(z+1)^{N/2}$

$$\frac{1}{N}\frac{\partial}{\partial z}\log p_{M_1} = \frac{1}{2}\frac{1}{z-1} + \frac{1}{2}\frac{1}{z+1} \Rightarrow \frac{z}{z^2-1} = g(z)$$

$$\Rightarrow z = \frac{1+\sqrt{1+4g^2}}{2g} \Rightarrow R = \frac{-1 \pm \sqrt{1+4g^2}}{2g}$$

$$\Rightarrow M = \tfrac{1}{2}(M_1+M_2) \quad \text{has} \quad R_M = R_{\frac{1}{2}(M_1+M_2)}$$

$$= 2R_{\frac{1}{2}M} = \frac{\sqrt{1+g^2}-1}{g}$$

$$\Rightarrow z = \sqrt{1+\tfrac{1}{g^2}} \Rightarrow \frac{1}{\pm\sqrt{z^2-1}} = g$$

$$\frac{1}{\pm\sqrt{z^2-1}} \;\downarrow\; \frac{1}{z\sqrt{1-\frac{1}{z^2}}} = g$$

$$\Rightarrow \rho(\lambda) = \frac{1}{\pi}\frac{1}{\sqrt{1-\lambda^2}} \qquad \lambda \in (-1,1)$$

Jacobi arcsine law

## 15.3.2 Sums of Uniform

$$M = U + OUO^T$$

$U$ is diag w $\lambda \sim \text{Unif}(-1,1)$

$$\rho_U = \tfrac{1}{2} \Rightarrow g(z) = \frac{1}{2}\int_{-1}^{1}\frac{d\lambda}{z-\lambda} = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right)$$

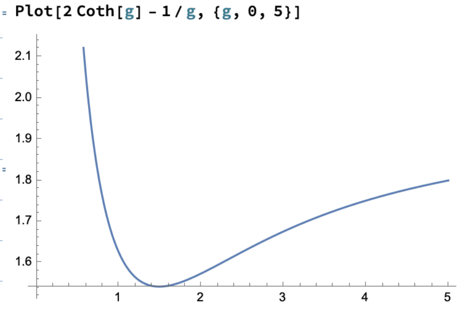
$$e^{2g} = \frac{z+1}{z-1} \Rightarrow z = \coth g$$

$$\Rightarrow R_U = \coth g - 1/g$$

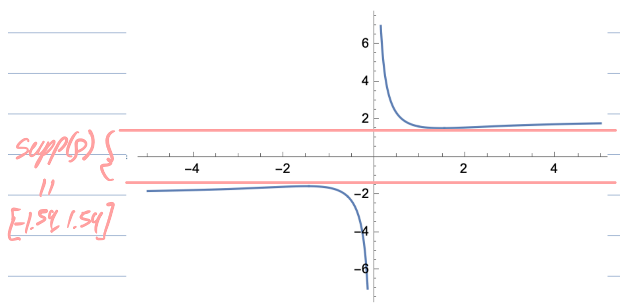
$$R_M = 2\coth g - 2/g$$

$$\Rightarrow z_M = 2\coth g - 1/g \quad \leftarrow \text{transcendental!}$$

Plot[2 Coth[g] - 1/g, {g, 0, 5}]



Plot[2 Coth[g] - 1/g, {g, -5, 5}]



## 15.4 Example: Multiplication

$$E = \sqrt{M_p}\, W_q\, \sqrt{M_p}$$

indep

$$S_{M_p} = 1 - pt \qquad S_{W_q} = \frac{1}{1+qt} \Rightarrow S_E = \frac{1-pt}{1+qt}$$

$$\Rightarrow \varsigma_E = \frac{t+1}{t}\,\frac{1+qt}{1-pt}$$ ⇒ quadratic in $t$

$$t_E = \frac{z-q-1 \mp \sqrt{(q+1-z)^2 - 4(q+zp)}}{2(q+zp)} \Rightarrow \rho_E = \frac{\sqrt{4(q+\lambda p) - (1+q-\lambda)^2}}{2\pi\lambda(q+p\lambda)}$$

edges at $\lambda_\pm = 1+2p-q \pm 2\sqrt{(1+p)(q+p)}$

$p \to 0 \Rightarrow 1-q \pm 2\sqrt{q} = (1\pm\sqrt{q})^2$ as expected

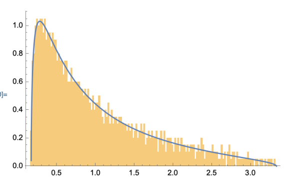
Ex 15.4.1 $p = 1/4 \Rightarrow q = 1/5$

a)

b)

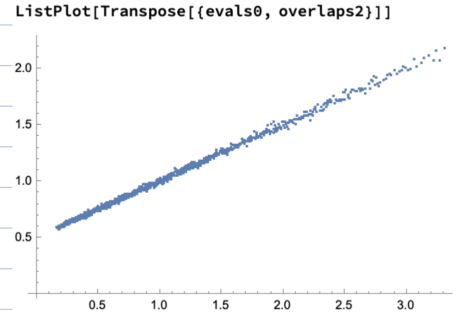
```
In[11]:= n = 1000; t = 5000;
M = RandomVariate[InverseWishartMatrixDistribution[t, (t - n - 1) IdentityMatrix[n]]];
n = 1000; t = 4000;
W = RandomVariate[WishartMatrixDistribution[t, 1/t IdentityMatrix[n]]];
E0 = MatrixPower[M, 1/2].W.MatrixPower[M, 1/2];

In[45]:= Tr[M] / n
Tr[M.M] / n
Tr[E0] / n
Tr[E0.E0] / n
evals0 = Eigenvalues[E0];
Show[Histogram[evals0, 200, "PDF"],
 Plot[Sqrt[4 (p λ + q) - (1 + q - λ)^2] / (2 π λ (p λ + q)) /. p → 1/4 /. q → 1/4, {λ, 0, 4}]]

Out[45]= 0.999972
Out[46]= 1.25124
Out[47]= 0.999669
Out[48]= 1.50124
```

Out[50]=




c)

```
overlaps = U.M.Transpose[U];
overlaps2 = Table[overlaps[[i, i]], {i, 1, 1000}];
ListPlot[Transpose[{evals0, overlaps2}]]
```



## 15.4.2 Free Product & Projectors

$$P_1 = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$ Projector $N \to N_1$

$P_2 : N \to N_2$

$P_1 P_2$ free product

$$g_{P_a} = \frac{N_a/N}{z-1} + \frac{\frac{N-N_a}{N}}{z} = \frac{q_a z + (1-q_a)(z-1)}{z(z-1)} = \frac{q_a + z - 1}{z(z-1)}$$

$$\Rightarrow t_{P_a} = \frac{q_a}{\zeta - 1}$$

$$\Rightarrow \zeta_{P_a} = \frac{q_a}{t} + 1 \qquad \Rightarrow S_{P_a} = \frac{t+1}{t+q_a}$$

$$\Rightarrow S_P = \frac{(t+1)^2}{(t+q_a)(t+q_b)}$$

$$\zeta_P = \frac{(t+q_a)(t+q_b)}{t(t+1)} \Rightarrow \text{quadratic}$$

$$\Rightarrow t = \frac{q_1+q_2-\zeta \overset{\oplus}{\pm} \sqrt{\zeta^2 - 2\zeta(q_1+q_2-2q_1q_2) + (q_1-q_2)^2}}{2(\zeta-1)}$$

$$\lambda_\pm = q_1+q_2-2q_1q_2 \pm 2\sqrt{q_1q_2(1-q_1)(1-q_2)}$$

$\lambda_- \geq 0 \qquad \lambda = 0$ iff $q_1 = q_2$

$$\Rightarrow g = \frac{1}{z} + \frac{t}{z}$$ Jacobi ensemble with $c_1 = \frac{q_{max}}{q_{min}}$ $\quad c_+ = \frac{1}{q_{min}}$

$$\Rightarrow \rho(\lambda) = \frac{\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}}{2\pi\lambda(1-\lambda)} + A_0\,\delta(z) + A_1(\delta(z-1))$$

$$A_0 = 1 - \frac{q_1 + q_2 - |q_1 - q_2|}{2} = 1 - \min(q_1, q_2)$$

$$A_1 = \frac{q_1+q_2-1 + |q_1+q_2-1|}{2} = \max(q_1+q_2-1, 0)$$

## 15.4.3 Jacobi Revisited

$$S_{W_{1,2}} = \frac{T_{1,2}^{-1}}{1+\frac{N}{T_{1,2}}t} = \frac{N^{-1}}{C_{1,2}+t} \qquad C_i = T_i / N_i$$

unnormalized

$$S_{W_{1,2}^{-1}} = N(C_{1,2}-t-1)$$

$$S_E = S_{W_1^{-1}} S_{W_2} = \frac{C_1-t-1}{C_2+t}$$

Hardest part is the shift

$$R(x) = \frac{1}{S_E(xR(x))}$$

$$(C_2+t)S_E = C_1-t-1$$

$$t = xR(x)$$

$$(C_2+xR) = (C_1-xR-1)R$$

$$S(t) = 1/R(x)$$

$$\Rightarrow C_2 + R_E(1+x-C_1+xR_E) = 0$$

$$C_2 + (R_{E+1}-1)(1-C_1+xR_{E+1}) = 0 \qquad \underbrace{S(t)}_{1/R} = \frac{1}{R(\underbrace{tS}_{x})} \Rightarrow R \to \frac{1}{S} \qquad x \to tS$$

$$\Rightarrow C_2 + (S_{E+1}^{-1}-1)(1-C_1+t) = 0$$

$$\Rightarrow (C_2+C_1-t-1)S_{E+1} + 1-C_1+t = 0$$

$$\Rightarrow S_{E+1} = \frac{1+t-C_1}{t+1-C_1-C_2}$$

$$\Rightarrow \boxed{S_J = (S_E(-t-1))^{-1} = \frac{t+C_1+C_2}{t+C_1}} \Rightarrow S(t) = \overbrace{\frac{t+1}{t}}^{\text{quad}} \frac{t+C_1}{t+C_1+C_2}$$

$$\Rightarrow t(\zeta) = \frac{1+c_1 - c_+\zeta \pm \sqrt{c_+^2\zeta^2 - 2(c_1c_+ + c_+ - 2c_1)\zeta + (c_1-1)^2}}{2(\zeta-1)}$$

$c_\pm = c_1 \pm c_2$

$x_1 = \frac{c_1}{c_1+c_2} \qquad x_2 = \frac{c_1c_2}{(c_1+c_2)^3} \quad \ldots$

From S:

$$R_J = \frac{x - c_+ - \sqrt{(c_+-x)^2 + 4c_1 x}}{2x}$$

$c_1 = c_2 = 1 \Rightarrow \quad S = \frac{t+2}{t+1} \qquad R = \frac{x-2-\sqrt{x^2+4}}{2x}$

For centered: $R_c(t) = 2R_c(2t) - 1$

# Chapter 16: Products of Many Random Matrices

16.1

$M_K: \quad A_K A_{K-1} \cdots A_1 A_1^T \cdots A_K^T$

$$S_{M_K} = \prod_{j=1}^{K} S_j(z) = S_1(z)^K$$

S-trans of $A_i A_i^T$

Assume $M_K \propto \mu^K$ w/ $\mu$ a random var

$$t_K = \int \frac{\mu^K}{z-\mu^K} p_\infty(\mu)\,d\mu = -\int \frac{1}{1-z\mu^{-K}} p_\infty(\mu)\,d\mu$$

Take $z = u^K \Rightarrow -\int \frac{1}{1-(\frac{u}{\mu})^K} p_\infty(\mu)\,d\mu$

$u > \mu \Rightarrow 0$
$u < \mu \Rightarrow$

$$t_K(u) = -\int_u^\infty p_\infty(\mu)\,d\mu =: -P_>(u) = -P_>(z^{1/K})$$

$$\zeta_K(t) = \left[ P_>^{-1}(-t) \right]^K$$

$$\Rightarrow S = \frac{t+1}{t\left[P_>^{-1}(-t)\right]^K} = S_1^{\,K}$$

$K \to \infty$ has $S_1 = \frac{1}{P_>^{-1}(-t)} \Rightarrow P_>(\mu) = -S_1^{-1}(1/\mu)$

$$\Rightarrow p_\infty = -P_>'(\mu) = \frac{\partial}{\partial\mu} S_1^{-1}(1/\mu)$$

For Wishart: $S_1 = \frac{1}{1+qx} \Rightarrow S^{-1}(\mu) = \frac{\mu^{-1}-1}{q}$

$$\Rightarrow -S^{-1}(1/\mu) = \frac{1-\mu}{q}$$

$$\Rightarrow p_\infty = 1/q \quad \text{for } \mu \in (1-q, 1)$$

Log Normal: $S_{LN}^{0} = e^{-a(z+1/2)} \Rightarrow S^{-1}(\mu) = -\frac{1}{a}\log\mu - \frac{1}{2}$

$$\Rightarrow P_>(\mu) = -S^{-1}\left(\tfrac{1}{\mu}\right) = -\tfrac{1}{a}\log\mu + \tfrac{1}{2} \Rightarrow p = -P_>'(\mu) = \frac{1}{a\mu} \qquad \mu \in \left(e^{-a/2}, e^{a/2}\right)$$

Lyapunov Exponent:

$$\Lambda := \lim_{K\to\infty} \frac{1}{K} \log M_K$$

e.g. for log-normal

$\Lambda$ is distributed uniformly on $(-a/2, a/2)$

So for the spectrum of the $K$ matrices is $K$-indep
$\Rightarrow$ leads to dist that depends on $S_1(z)$ explicitly
$\Rightarrow$ Non-universal

Instead assume $AA^T = \left(1+\frac{a}{2K}\right)\mathbb{1} + \frac{B}{\sqrt{K}}$ $\qquad \tau(B)=0$
$\tau(B^2) =: b$

$$\Rightarrow \quad S_1(z) = 1 - \frac{a}{2K} - \frac{b}{K} z + o(K^{-1})$$

$$\Rightarrow \quad S_{M_K}(z) = \left(1 - \frac{a}{2K} - \frac{b}{K} z\right)^K \to e^{-\frac{a}{2} - bz}$$ } Multiplicative CLT

$b = a \Rightarrow$ free log-normal

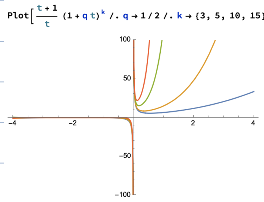
## Ex 16.1.1 Edges of Spectrum

a) $S_1 = \dfrac{1}{1+qx} \Rightarrow S_M = (1+qt)^{-K}$

$$\Rightarrow \zeta_M = \frac{t+1}{t}(1+qt)^K$$

b)

```
Plot[(t + 1)/t (1 + q t)^k /. q → 1/2 /. k → {3, 5, 10, 15} // Evaluate, {t, -4, 4}, PlotRange → 100]
```




c) $\zeta_M' = \dfrac{(1+qt)^{K-1}\left(-1+qt(-1+k+kt)\right)}{t^2}$

$$\frac{-q^2+3q}{2Kq} \qquad \frac{q^2-q}{2Kq}$$

$$\zeta' = 0 \Rightarrow t = \frac{q(1-K) \pm \sqrt{q}\sqrt{4Kq - 2Kq + K^2 q}}{2Kq} = \frac{-qK \pm Kq}{2qK} + \frac{q \pm (2-q)q}{2Kq}$$

$\zeta = \qquad \Rightarrow 0$

d) $\displaystyle \lim_{K\to\infty} \frac{1}{K} \log \lambda_\pm = \cancel{*} + \begin{cases} 1 \\ 1-q \end{cases}$

## 16.2 Free Log-Normal

$$S_{LN} = \exp(-a/2 - bz)$$

Product of lognormals $AB = C$ is lognormal

$a_A + a_B = a_C$

$b_A + b_B = b_C$

$$S_{2N} \approx e^{-a/2}\left[1 - bz + \frac{b^2 z^2}{2}\right]$$

$$x_1 = e^{a/2}$$
$$x_2 = e^{a} b$$
$$x_3 = e^{2a}\,\frac{3b^2}{2}$$

When $b=a$ $\quad S[-z-1]^{-1} = e^{a/2} e^{-a(z+1)} = e^{-\frac{a}{2}-az}$

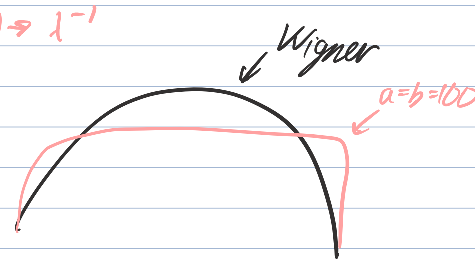
$$\zeta(t) = \frac{t+1}{t}\, e^{\frac{a}{2}+at}$$

$$\zeta' = 0 \Rightarrow t = \frac{-1 \pm \sqrt{1+4/a}}{2}$$

$$\Rightarrow \lambda_+ = e^{\sqrt{a+a^2/4}}\left(\sqrt{\frac{a}{4}} + \sqrt{1+\frac{a}{4}}\right)^2 = 1/\lambda_-$$

$a=0 \Rightarrow \lambda_+ = \lambda_- = 0$ $\quad$ symmetric in $\lambda \to \lambda^{-1}$

$l = \log \lambda$ has even density




$a \neq b \Rightarrow l$ is shifted by $\frac{a-b}{2}$

## 16.3 Multiplicative Dyson Brownian Motion

$$M_{n+1} = \sqrt{M_n}\left[\left(1+\frac{a\epsilon}{2}\right)\mathbb{1} + \sqrt{\epsilon}\, B_n\right]\sqrt{M_n}$$

Noise

$\tau(B_n) = 0$

$$\lambda_{i,n+1} = \lambda_{i,n}\left(1 + \frac{a\epsilon}{2} + \sqrt{\epsilon}\, v_{i,n}^T B_n v_{i,n}\right) + \epsilon \sum_{j\neq i} \frac{\lambda_{i,n}\lambda_{j,n}(v_{i,n}^T B_n v_{j,n})^2}{\lambda_{i,n} - \lambda_{j,n}}$$

Since $M, B$ free $\quad \tau(B) = 0 \Rightarrow \mathbb{E}[v_i^T B v_i] = 0$

$$\mathbb{E}[(v_i B v_j)^2] = \frac{b}{N} \qquad b = \tau(B_n^2)$$

Take $\quad \epsilon = dt \Rightarrow \dfrac{d\lambda_i}{dt} = \dfrac{a}{2}\lambda_i + \dfrac{b}{N}\sum_{j\neq i}\dfrac{\lambda_i\lambda_j}{\lambda_i - \lambda_j} + \sqrt{\dfrac{b}{N}}\,\lambda_i \xi_i$

*

$$g(z,t)=\frac{1}{N}\sum_j \frac{1}{z-\lambda_j(t)} \Rightarrow \frac{dg}{dt}=\frac{1}{N}\sum_j \frac{1}{(z-\lambda_j)^2}\frac{d\lambda_j}{dt}=-\frac{1}{N}\frac{\partial}{\partial z}\sum_j \frac{1}{z-\lambda_j}\frac{d\lambda_j}{dt}$$

$$\underbrace{-\frac{a}{2N}\sum_j \frac{\lambda_j}{z-\lambda_j}}_{-\frac{a}{2}zg(z)} - \frac{b}{N^2}\sum_{\substack{i,j\\ j\neq i}} \frac{\lambda_i\lambda_j}{(\lambda_i-\lambda_j)(z-\lambda_i)} - \underbrace{\sqrt{\frac{b}{N}}\frac{1}{N}\sum_j \frac{\lambda_j}{z-\lambda_j}\xi_j}_{\mathbb{E}=0}$$

$$\frac{1}{N^2}\frac{1}{2}\sum_{i\neq j}\left[\frac{\lambda_i\lambda_j}{(z-\lambda_i)(\lambda_i-\lambda_j)}+\frac{\lambda_i\lambda_j}{(z-\lambda_j)(\lambda_j-\lambda_i)}\right]=\frac{1}{2}\frac{1}{N^2}\sum_{i\neq j}\frac{\lambda_i\lambda_j}{(z-\lambda_i)(z-\lambda_j)}=\frac{1}{2}\left(\frac{1}{N}\sum_i \frac{\lambda_i}{z-\lambda_i}\right)^2$$

$$-\sum_j \frac{\lambda_j}{z-\lambda_j}=\sum_j 1-\sum \frac{z}{z-\lambda_j}$$

$$\Rightarrow \frac{1}{2}(1-zg(z))^2=\cancel{\frac{1}{2}}-zg+\frac{z^2g^2}{2}$$

$$bzg-\frac{b}{2}z^2g^2$$

$$\Rightarrow \frac{\partial g}{\partial t}=\frac{1}{2}\frac{\partial}{\partial z}\left[(2b-a)zg-bz^2g^2\right]$$

$$h=e^l g(e^l,t)+\frac{a}{2b}-1 \qquad e^l g(e^l,t)=h-\frac{a}{2b}+1$$

$$\Rightarrow \frac{\partial h}{\partial t}=e^l\frac{\partial g}{\partial t}(e^l)$$

$$=\frac{e^l}{2}\frac{\partial}{\partial e^l}\left[(2b-a)e^l g-be^{2l}g^2\right]$$

$$=\frac{1}{2}\frac{\partial}{\partial l}\left[(2b-a)(h-\cancel{\tfrac{a}{2b}+1})-b\underbrace{(h-\tfrac{a}{2b}+1)^2}_{bh^2+(2b-a)h+\cancel{\text{const}}}\right]$$

$$=-\frac{1}{2}\frac{\partial}{\partial l}(bh^2)$$

$$=-bh\frac{\partial h}{\partial l} \quad \leftarrow \text{after } t'=bt\text{, Burgers'}$$

$\Rightarrow$ Method of characteristics $\qquad g(z,0)=(z-1)^{-1}$

$$h(l,t)=h_0(l-bt\,h(l,t)) \qquad h_0(l):=h(l,0)=\frac{1}{1-e^{-l}}+\frac{a}{2b}-1$$

$$\Rightarrow g(z,t)=\frac{1}{z-\exp[t(bzg+\frac{a}{2}-b)]}$$

$$\zeta = \frac{t+1}{t} \exp\left( a/2 + bt \right)$$

$$t = zg - 1 \Rightarrow z = \frac{zg}{zg-1} e^{a/2 - b + bzg} \Rightarrow zg - 1 = g e^{a/2 - b + bzg} \Rightarrow g = \left(z - e^{a/2 - b + bzg}\right)^{-1}$$

same as $g(z, t=1)$

## 16.4 The Matrix Kesten Problem

$$Z_{n+1} = z_n (1 + Z_n)$$

$$z_n = 1 + \varepsilon m + \sqrt{\varepsilon}\, \sigma \eta_n$$

$$Z_n = U_n / \varepsilon$$

$$U_{n+1} = \varepsilon \left(1 + \varepsilon m + \sqrt{\varepsilon} \sigma \eta_n\right)\left(1 + U_n/\varepsilon\right)$$

$$= U_n + \varepsilon m U_n + \sqrt{\varepsilon} \sigma \eta_n U_n + \varepsilon$$

$$\rightarrow \frac{dU}{dt} = 1 + \underbrace{mU}_{\sigma^2/2} + \sigma \eta U$$

part of the noise (multiplicative)

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial U}\left[(1 + mU)P\right] + \frac{\sigma^2}{2} \frac{\partial^2}{\partial U^2}\left[U^2 P\right]$$

$$\frac{\partial P}{\partial t} = 0 \Rightarrow (1 + \underbrace{m}_{-\hat{m}} U)P = \frac{\sigma^2}{2} \frac{\partial}{\partial U}(U^2 P)$$

$$\Rightarrow P_{eq} \propto \frac{e^{-2/\sigma^2 U}}{U^{1+\mu}} \qquad \mu = 1 + \frac{2\hat{m}}{\sigma^2}$$

inverse $\Gamma$

Power law tail $U^{-1-\mu}$

w/ non-universal exponent

Now for matrices:

$$U_{n+1} = \varepsilon \sqrt{1 + \frac{U_n}{\varepsilon}} \left( (1 + m\varepsilon)\mathbb{1} + \sqrt{\varepsilon}\, \sigma B \right) \sqrt{1 + \frac{U_n}{\varepsilon}}$$

$$\Rightarrow U_{n+1} - U_n = \varepsilon\left(\mathbb{1} + m U_n\right) + \sigma \sqrt{\varepsilon} \sqrt{U_n}\, B \sqrt{U_n}$$

$$\Rightarrow \lambda_{i,n+1} - \lambda_{i,n} = \varepsilon\left(1 - \hat{m} \lambda_{i,n}\right) + \underbrace{\sqrt{\varepsilon}(\cdots)}_{E=0} + \varepsilon \sum_{j \neq i} \frac{\lambda_{i,n} \lambda_{j,n}}{\lambda_{i,n} - \lambda_{j,n}} \underbrace{\left(v_{i,n}^T B\, v_{j,n}\right)^2}_{\sigma^2/N}$$

$$\Rightarrow \frac{d\lambda_i}{dt} = 1 - \hat{m} \lambda_i + \frac{\sigma^2}{N} \sum_{j \neq i} \frac{\lambda_i \lambda_j}{\lambda_i - \lambda_j}$$

$\Leftarrow$ as in Mult Dyson motion

as before

$$\Rightarrow \frac{\partial g}{\partial t} = \frac{\partial}{\partial z}\left[-g + (\sigma^2+\hat{m})\, zg - \frac{1}{2}\sigma^2 z^2 g^2\right]$$ ✓

$$\Rightarrow \left(1 - z(\sigma^2+\hat{m})\right) g + \frac{1}{2}\sigma^2 z^2 g^2 + C = 0$$

$zg \to 1$ as $z \to \infty \Rightarrow C = \frac{1}{2}\sigma^2 + \hat{m}$

$$\Rightarrow g = \frac{1}{\sigma^2 z^2}\left[(\sigma^2+\hat{m})z - 1 \overset{+}{-} \sqrt{\hat{m}^2 z^2 - 2(\sigma^2+\hat{m})z + 1}\right]$$

$$\lambda_\pm = \frac{\sigma^2 + \hat{m} \pm \sqrt{\sigma^2(\sigma^2+2\hat{m})}}{\hat{m}^2}$$

$\hat{m} \to 0 \Rightarrow$ unbounded spec

else truncated at $2\sigma^2/\hat{m}^2$

Universal exponent $\mu = 1/2$ at $m=0$ Unlike scalar case!

$x = \frac{2}{(\sigma^2+\hat{m})\lambda} \Rightarrow$ Marčenko-Pasteur $q = \frac{\sigma^2}{\sigma^2+2\hat{m}} = \frac{1}{\mu} < 1$

# Chapter 17: Sample Covariance Matrices

Sample Cov: $E = \frac{1}{T} HH^T \qquad H \in \mathbb{R}^{N\times T}$

$T \gg N \Rightarrow E \to C \leftarrow$ true Cov

$C_{ij}$ are spatial correlations

The $T$ samples yield temporal correlations

Care about $\lambda_k$ of $E$ and singular values $s_k$ of $H$ $\quad s_k = \sqrt{T\lambda_k}$

## 17.1 Spatial Correlations

$H \sim N(0, C)$

$\Rightarrow E$ is Wishart with $cov = C$

$$E = \sqrt{C}\, W_q \sqrt{C}$$

$$\Rightarrow S_E = \frac{S_C(t)}{1+qt}$$

Recall: $t_{AB} = t_A(z S(t_{AB}(z)))$

$\Rightarrow t_E = t_c(Z(z)) \qquad Z(z) = \frac{z}{1+qt_E(z)}$

$z g_E(z) = Z g_c(Z) \qquad Z = \frac{z}{1-q+qzg_E(z)}$

$$\Rightarrow g_E = \int \frac{\rho_c(\mu)d\mu}{z-\mu(1-q+qzg_E(z))}$$

True $\forall H$ with finite 2nd moment $\Rightarrow$ Universality

$q \to 0 \Rightarrow g_E = g_c \qquad \rho_E = \rho_c$

Wishart

$C = \mathbb{1} \Rightarrow g_c = \frac{1}{z-1} \Rightarrow z g_E = \frac{z}{z-1+q-qzg_E} \Rightarrow \frac{1}{g_E} = z+q-qzg_E$

$$t_E \to \sum_{k=1}^{\infty} \tau(E^k) z^{-k} \Rightarrow Z(z) = \frac{z}{1+q\sum_{k=1}^{\infty} \tau(E^k) z^{-k}}$$

$$\Rightarrow t_C(Z(z)) \to \sum_{k=1}^{\infty} \frac{\tau(C^k)}{z^k}\left(1+q\sum_{l=1}^{\infty}\frac{\tau(E^l)}{z^l}\right)^k = \sum_{k=1}^{\infty}\frac{\tau(E^k)}{z^k}$$

$$\tau(E) = \tau(C)$$
$$\tau(E^2) = \tau(C^2)+q$$
$$\tau(E^3) = \tau(C^3) + 3q\tau(C^2) + q^2$$

$q < 0 \Rightarrow E$ invertible

$$g_E(z) = \sum_{k=1}^{\infty} z^{k-1}\tau(E^{-k}) \qquad Z = \frac{z}{1-q+q\sum_k \tau(E^{-k}) z^k}$$

$$\Rightarrow \sum_{k=1}^{\infty} z^k \tau(E^{-k}) = \sum_k \tau(C^{-k})\left(\frac{z}{1-q}\right)^k\left[1-\frac{q}{1-q}\sum_{l=1}^{\infty}\tau(E^{-l})z^l\right]^{-k}$$

$$\Rightarrow \tau(E^{-1}) = \frac{\tau(C^{-1})}{1-q} \quad ✓$$

related to gen err

$$\tau(E^{-2}) = \frac{\tau(C^{-2})}{(1-q)^2} + \frac{q\,\tau(C^{-1})^2}{(1-q)^3} \quad ✓$$

Ex 12.11 EMA-SCM

$$E(t) = \gamma_c \sum_{t'=-\infty}^{t} (1-\gamma_c)^{t-t'} x_{t'} x_{t'}^T$$

$$\Rightarrow E(t) = (1-\gamma_c)E(t-1) + \gamma_c x_t x_t^T \leftarrow T=1 \text{ Wishart} \Rightarrow q=N$$

$$E \overset{\text{in law}}{=} (1-\gamma_c)E + \gamma_c x x^T$$

a) By R trans: $R_E[x] = (1-\gamma) R_E[(1-\gamma)x] + \gamma \underbrace{R_{W,q=N}[\gamma x]}_{\gamma_c(1-N\gamma_c x)^{-1}}$

b) Let $\gamma_c = 1/\tau_c$ $q := N/\tau_c$ fixed

$$\Rightarrow R_E[x] = (1 - \tfrac{q}{N}) R_E[(1-\tfrac{q}{N})x] + \tfrac{q}{N}(1-qx)^{-1}$$

$$\cancel{R_E[x]} = \cancel{R_E[x]} - \tfrac{q}{N} R_E[x] - \tfrac{qx}{N} R_E'(x) + \tfrac{q}{N}\tfrac{1}{1-qx}$$

$$\Rightarrow R_E[x] = x\, R_E'[x] + \frac{1}{1-qx}$$

c) $\tau(xx^T) = 1 \Rightarrow \tau(E) = 1 \Rightarrow R[0] = 1$

$$R_E[x] = -\frac{\log(1-qx)}{qx}$$

$$\Rightarrow = 1 + \frac{qx}{2} + \frac{(qx)^2}{3} + \cdots$$

$$\Rightarrow x_2 = q^2/3$$

d) $\mathfrak{z}(g) = R[g] + 1/g$ $+ \cdots$

FINISH

## 17.2 Temporal Correlations

$T$ samples are correlated $\Rightarrow T_{eff} < T$

Assume $C = \mathbb{1}$ (White spatial correlates)

$\Rightarrow E[H_{it} H_{js}] = \delta_{ij} \underbrace{K_{ts}}_{T\times T \text{ cov matrix}}$

assume $\tau(K)=1$

$\Rightarrow H = H_0 K^{1/2}$ $H_0$ white

$\Rightarrow E := \frac{1}{T} HH^T = \frac{1}{T} H_0 K H_0^T$ ← not quite a free prod of $K$ & a Wishart

$F := \frac{1}{N} H^T H = \frac{1}{N} K^{1/2} H_0^T H_0 K^{1/2} = K^{1/2} W_{q^{-1}} K^{1/2}$ ← free prod

$\Rightarrow S_F = \frac{S_K}{1+t/q}$

Recall (Eq 4.5) $g_F(z) = q^2 g_E(qz) + \frac{1-q}{z}$

$\Rightarrow t_F(z) = z q^2 g_E(qz) - q$

$= q t_E(qz)$

$\Rightarrow \zeta_E = q \zeta_F(qt)$ $\Rightarrow$ $\frac{t+1}{t\zeta_E(t)} = \frac{t+1}{qt\, \zeta_F(qt)}$

$\Rightarrow S_E = q^{-1} S_F(qt)$ $\Rightarrow S_E = \frac{t+1}{1+qt} S_F(qt)$

$\Rightarrow S_E(t) = \frac{S_K(qt)}{1+qt}$

$\Rightarrow \zeta_E = \frac{t+1}{t} \frac{1+qt}{S_K(qt)} = q(t+1)\zeta_K(qt)$

$t_K\left[\frac{z}{q(t_E+1)}\right] = q t_E(z)$ $t_E(z) \overset{z\to 0}{=} -1 - z\,\tau(E^{-1}) + O(z^2)$ by def'n

$\Rightarrow t_K\left[-\frac{1}{q\tau(E^{-1})}\right] = -q$ $\Rightarrow \tau(E^{-1}) = -\frac{1}{q\zeta_K(-q)}$

## 17.2.2 Exponential Correlations

$K_{ts} = a^{|t-s|}$ $\qquad \tau = |1/\log a| \approx \frac{1}{1-a}$

Toeplitz

$$K = \begin{pmatrix} 1 & a & a^2 & \cdots & a^{T-1} \\ a & 1 & a & & \vdots \\ \vdots & & 1 & & \vdots \\ & & & \ddots & \\ a^{T-1} & \cdots & & & 1 \end{pmatrix}$$

Take $T \to \infty$

$$\sum_{t=-\infty}^{\infty} K_{ts} e^{2\pi i x s} = e^{2\pi i x t} \sum_{\ell=-\infty}^{\infty} a^{|\ell|} e^{2\pi i x \ell}$$

$\tau_c := \frac{1}{1-a}$

First modify $K \to \tilde{K}$ so $|s-t|$ lies on a circle

$$\tilde{K}_{ts} = a^{-\min[|t-s|, |t-s+T|, |t-s-T|]}$$

Circulant

Only changes bottom right & top left change drastically

$\Rightarrow [v_k]_\ell = e^{2\pi i k \ell / T} \qquad 0 \le k \le T/2$

Each $v_k$ is really 2 eves: real & im of $v_k$

Except $v_0 = \mathbb{1}$, $v_{T/2} = (+ - + - \cdots)$

$$k=0 \Rightarrow \lambda_0 = \lambda_+ = 1 + 2\sum_{k=1}^{T/2-1} a^k + a^{T/2} \approx \frac{1+a}{1-a}$$

$$\lambda_{T/2} = \lambda_- = 1 + 2\sum_{k=1}^{T/2-1} (-a)^k + (-a)^{T/2} \approx \frac{1-a}{1+a} = 1/\lambda_+$$

$\lambda_+ = 2\tau_c - 1$

let $x_k = \frac{2k}{T} \in [0,1]$ index the eigs

$$\lambda(x) = \frac{1-a^2}{1+a^2+2a\cos\pi x} \Leftarrow \text{SHOW}$$

For general $K_{ts} = K(|t-s|)$ $\qquad \lambda = 1 + 2\sum_{\ell=1}^{\infty} K(\ell) \cos \pi x \ell$

returning: $\qquad t_K(z) = \int_0^1 \frac{1-a^2 \;\; = \frac{\lambda}{z-\lambda}}{z(1+a^2+2a\cos\pi x) - (1-a^2)}\, dx$

Can show $t_K = \dfrac{1}{\sqrt{z-\lambda_-}\sqrt{z-\lambda_+}} \Rightarrow p(\lambda) = \dfrac{1}{\pi\lambda}\dfrac{1}{\sqrt{(\lambda_+-\lambda)(\lambda-\lambda_-)}}$

$\int_0^1 \frac{dx}{c-d\cos\pi x} = \frac{1}{\sqrt{c-d}\sqrt{c+d}}$

$\lambda_- < \lambda < \lambda_+$

$t^2 \zeta_K^2 - 2bt^2\zeta_K + t^2 - 1 = 0 \qquad b = \dfrac{1+a^2}{1-a^2}$

$\Rightarrow \zeta_K = \dfrac{b\cancel{t^2} + \sqrt{(b^2-1)\cancel{t^4} + t^{\cancel{2}-2}}}{\cancel{t^2}}$

$\Rightarrow S_K = \dfrac{t+1}{bt + \sqrt{(b^2-1)t^2+1}} \approx 1-(b-1)t + O(t^2)$

$\Rightarrow \tau(K) = 1$

$\kappa_2(K) = b-1 = \dfrac{2a^2}{1-a^2}$

Using $S_E = \dfrac{S_K(qt)}{1+qt} = \left(bqt + \sqrt{(b^2-1)(qt)^2+1}\right)^{-1}$

$\Rightarrow \zeta_E = \left(bqt + \sqrt{(b^2-1)(qt)^2+1}\right)\dfrac{t+1}{t}$

4th order in $t$ to invert :(

$\tau(E^{-1}) = \dfrac{-(-q)}{q\,b(-q) + q\sqrt{1+(b^2-1)q^2}}$

$= \dfrac{1}{\sqrt{1+(b^2-1)q^2} - qb} =: \dfrac{1}{1-q^*} \qquad q^* =: N/T^*$

effective length of time series

$a \to 0 \Rightarrow b=1 \Rightarrow q = q^*$

$a \to 1 \Rightarrow b = \infty \Rightarrow \frac{1}{b}\ \frac{1}{q-q} \to \infty \Rightarrow q = 1$

$q^* = q(1+2a^2+\cdots)$ for $a$ small

$b \gg 1 \Rightarrow a \to 1$, long-range correlations

$S_E \approx (2bqt)^{-1}$ $\quad S_E$ depends only on $bq$ jointly

Define $qb := \sigma^2$ Take $b \to \infty$, $q \to 0$, $qb = \sigma^2$ fixed

$\Rightarrow S(t) = \dfrac{1}{\sqrt{1+\sigma^4 t^2} + \sigma t}$

$\Rightarrow R(z) = \dfrac{1}{\sqrt{1-\sigma^2 z}} = 1 + \sigma^2 z + \frac{3}{2}\sigma^4 z^2 + O(z^3)$

$\Rightarrow \kappa_2 = \sigma^2$

$\kappa_3 = \frac{3}{2}\sigma^4$ ← for Wishart $\kappa_3 = \sigma^4$

Unlike MP, no support or Dirac mass @ 0 even for $\sigma^2 > 1$

$\lambda_\pm$ is 4th order in $z$ here

Intuitive understanding:

$N$ Ornstein-Uhlenbeck process w $\tau_c$
recorded over $T \gg \tau_c$
# observations $= T/\Delta$ $\Delta =$ window size
$\Rightarrow N \times N$ sample cov
$\Delta \gg \tau_c \Rightarrow$ samples uncorrelated
$\Rightarrow$ MP with $q = N\Delta/T$
$\Delta \ll \tau_c \Rightarrow$ samples correlated
$E$ depends not on $\Delta$ but on $\tau_c$
$\Rightarrow \sigma^2 = qb = N\tau_c/T$ } Not MP

## 17.2.3 Spatial & Temporal Correlations

$$E = \frac{1}{T} H H^T = \frac{1}{T} C^{1/2} H_0 K H_0^T C^{1/2}$$

Get: $S_E(t) = \dfrac{S_C(t)\, S_K(qt)}{1+qt}$

$\Rightarrow \varsigma_E = \dfrac{1+t}{t\, S_C(t)} \dfrac{1+qt}{S_K(qt)}$

$= q\, \varsigma_C(t)\, \varsigma_K(qt)$ $\Rightarrow q t_E(z) = t_K\left( \dfrac{z}{q\, t_E(z)\, \varsigma_C(t_E(z))} \right)$ *

$C = \mathbb{1} \Rightarrow \varsigma_C = \frac{1+t}{t} \Rightarrow q t_E = t_K\left(\frac{z}{q(1+t_E(z))}\right)$ as before

Now specialize to $K_{st} = q^{|s-t|}$

$$t_K(z) = \frac{-1}{\sqrt{z^2 - 2zb + 1}}$$

$$\Rightarrow \frac{1}{q^2 t_E^2} = \frac{z^2}{q^2 t_E^2 \zeta_c(t_E)^2} - \frac{2zb}{q t_E \zeta_c(t_E)} + 1$$

$$\Rightarrow \zeta_c(t_E)^2 = z^2 - 2z\underbrace{bq}_{\approx \sigma^2}\, t_E \zeta_c(t_E) + \cancel{q^2 t_E^2 \zeta_c(t_E)^2} \rightarrow 0 \text{ as } q \to 0$$

$$C = \mathbb{1} \Rightarrow \left(\frac{1+t}{t}\right)^2 = z^2 - 2z\sigma^2\left(\frac{1+t}{t}\right)$$ $\Leftarrow$ Same Eq for $t$ as in previous subsection

$C = W^{-1} \Rightarrow \zeta_c(t) = \frac{t+1}{t(1-pt)} \Rightarrow$ 4th order eq for $t$

$z \to 0 \Rightarrow$ by $*$ $\quad -q = t_K\left(\frac{z}{q(-1)\zeta_c(-1+zg_E(0))}\right) \Rightarrow \zeta_K(-q) = \frac{\tau(C^{-1})}{-q\,\tau(E^{-1})}$

$t_E(0) = -1$

$1 + t_E(z) = zg_E(0) = z\tau(E^{-1}) \qquad \left(1+\frac{1}{t}\right)\frac{1}{\zeta}$

$\zeta(-1 + zg_E(0))$

$$\boxed{\Rightarrow \tau(E^{-1}) = -\frac{\tau(C^{-1})}{q\,\zeta_K(-q)}}$$

$$t_c(z) = -1 + zg_c(z) \approx -1 + z\tau(C^{-1})$$

$$\Rightarrow \zeta(-1+\varepsilon) = \frac{\varepsilon}{\tau(C^{-1})}$$

For matrix w/ purely spatial correlations we had

$$\tau(E^{-1}) = \frac{\tau(C^{-1})}{1-q}$$

$$\boxed{\Rightarrow q^* = 1 + q\,\zeta_K(-q)}$$

## Ex 17.2.1 Futility of Oversampling

$T$ indep obs, true corr $= C$

Repeat them $m$ times $\to$ $mT$ columns

a) Temporal correlation $=$ $$\left.\begin{pmatrix} \overbrace{\blacksquare}^{m\times m} & & 0 \\ 0 & \blacksquare & \\ & & \blacksquare \end{pmatrix}\right\} T \text{ blocks} \Rightarrow \begin{array}{l} T \text{ eigs} = m \\ (T-1)m \text{ eigs} = 0 \end{array}$$

b) $t_K = \frac{1}{Tm}\left(\frac{Tm}{z-m} + \cancel{\frac{(T-1)0}{z}}\right) = \frac{1}{z-m}$

c) $\zeta = \frac{1+mt}{t} \Rightarrow S_K = \frac{1+t}{1+mt}$

d) $q_m = N/mT$ naively but

$$S_E = S_C(t)\, \frac{S_K(q_m t)}{1+q_m t} = \frac{S_C(t)}{1+q_m m t} = \frac{S_C(t)}{1+qt}$$ $m$ cancels!

## 17.3 Time-dependent variance

$N$ time series are heteroscedastic

$x_i^t = \sigma_t H_{it}$ $\quad \mathbb{E}[H_{it} H_{js}] = \delta_{ts} C_{ij}$

time-dep

$\Rightarrow E = \sum_{t=1}^{T} P_t \;, \quad P_t := \frac{1}{T}\sigma_t^2 H_t H_t^T$

$\Rightarrow R_E(g) = \sum_t R_t(g)$ ← need $1/N$ corrections since $T$ terms!

$$g_t = \frac{1}{N}\left[\frac{N-1}{z} + \frac{1}{z-q\sigma_t^2}\right] = \frac{1}{z} + \frac{1}{N}\frac{q\sigma_t^2}{z(z-q\sigma_t^2)}$$

$$\Rightarrow \mathfrak{z}_t(g) = \frac{1}{g} + \frac{1}{N}\frac{q\sigma_t^2}{1-q\sigma_t^2 g}$$ $\frac{q}{N} = \frac{1}{T}$

$$\Rightarrow R_E = \frac{1}{T}\sum_t \frac{\sigma_t^2}{1-q\sigma_t^2 g}$$ encode $s = \sigma^2$

$$\to \int_0^\infty \frac{s\, P(s)\, ds}{1-qsg}$$

$P(s) = \delta(s-1) \Rightarrow \frac{1}{1-qg} = R_{Wishart}$

Generally $R_E = t_s\left(\frac{1}{qg}\right)$

When $C \neq \mathbb{1}$ can write $S_E = S_C(t)\, S_{\tilde{E}}(t)$

View $\sigma_t$ as diagonal temporal cov w/ entries drawn from $P(s)$

$$\Rightarrow S_E = \frac{S_s(qt)}{1+qt} \Rightarrow S_{\tilde{E}} = \frac{S_{\varepsilon}(t)\, S_s(qt)}{1+qt}$$

When $P(s)$ is inv-gamma & $x^t$ is Gaussian $\to \sigma_t x^t$ is a multivariate Student $t$

## 17.4 Empirical Cross-Covariance

$x^t \in \mathbb{R}^{N_1}$
$y^t \in \mathbb{R}^{N_2}$

$$E_{xy} = \frac{1}{T}\sum_{t=1}^{T} x^t (y^t)^T \in \mathbb{R}^{N_1 \times N_2}$$

Assume "true" $\mathbb{E}[xy^T] = 0$

What is the singular value spec of $E_{xy}$

$q_1 = N_1/T \quad q_2 = N_2/T$, take asymptotic limit

Consider $E_{xy} E_{xy}^T \leftarrow N_1 \times N_1$

Def $\hat{E}_x = x^T x \in \mathbb{R}^{T\times T}$
$\hat{E}_y = y^T y \in \mathbb{R}^{T\times T}$

$E_{xy}$ has same nonzero $\lambda$ as $\hat{E}_x \hat{E}_y$

Consider first "sample-normalized" PCs

$x \to \tilde{x} \Rightarrow \hat{E}_{\tilde{x}}$ has $N_1$ eigs $=1$ $T - N_1 = 0$
$y \to \tilde{y} \Rightarrow \hat{E}_{\tilde{y}}$ has $N_2$ eigs $=1$, $T - N_2 = 0$

$\Rightarrow$ Singular spectrum is $\rho(s) = \max(q_1+q_2-1, 0)\,\delta(s-1) + \mathrm{Re}\, \frac{\sqrt{(s^2-\gamma_-)(\gamma_+-s^2)}}{\pi s(1-s^2)}$

$\gamma_\pm = q_1 + q_2 - 2q_1q_2 \pm 2\sqrt{q_1 q_2 (1-q_1)(1-q_2)}$

$0 \le \gamma_\pm \le 1$

$\Rightarrow$ $s$'s are linear combinations of $x$s correlated into linear combinations of $y$s

$T \to \infty$ at fixed $N_1, N_2 \Rightarrow$ all $s \in [\,|\sqrt{q_1} - \sqrt{q_2}|, \sqrt{q_1} + \sqrt{q_2}\,] \Rightarrow s \sim T^{-1/2} \to 0$

Gives rise to statistical tests for cross-correlations

# Chapter 18: Bayesian Estimation

## 18.1.2 A Simple Estimation Problem

$y = x + \varepsilon$

$P(y|x) = P_\varepsilon(y-x)$

Assume $\varepsilon$ Gaussian $\Rightarrow P(y|x) = \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left[-\frac{(y-x)^2}{2\sigma_n^2}\right]$

$\Rightarrow P(x|y) \propto P_0(x) \exp\left[-\frac{x^2}{2\sigma_n^2} + \frac{2xy}{2\sigma_n^2}\right]$

$P_0$ is Gaussian $N(x_0, \sigma_s^2) \Rightarrow P(x|y) = N(\hat{x}, \sigma^2)$

$\hat{x} = x_0 + r(y - x_0) = (1-r)x_0 + ry$

$\sigma^2 = (\sigma_n^{-2} + \sigma_s^{-2})^{-1} = r\sigma_n^2 \qquad r = \frac{\sigma_s^2}{\sigma_n^2 + \sigma_s^2}$ "SNR" (not really)

Linear Shrinkage Estimator

$\mathrm{Var}\,\hat{x} = r^2 \,\mathrm{Var}\,y = r^2[\mathrm{Var}\,y|x + \mathrm{Var}\,x]$

$= r^2(\sigma_n^2 + \sigma_s^2) = \frac{\sigma_s^4}{\sigma_n^2 + \sigma_s^2} \le \sigma_s^2 \le \mathrm{Var}\,x$

$\Rightarrow$ estimator's variance is smaller than the truth!

Ex 18.1.1 Optimal Affine Estimator

$\hat{x} = ay + b$

min over $a, b$

a) If $Ex = Ey = 0 \Rightarrow \mathbb{E}[(x-\hat{x})^2] = \mathbb{E}[x^2] + \mathbb{E}[(\hat{x})^2] - 2\,\mathbb{E}\,x\hat{x}$

$= \sigma_x^2 + a^2\sigma_y^2 + b^2 - 2a\sigma_{xy}^2$

b) $\partial_a = 0 \Rightarrow a = \sigma_{xy}^2/\sigma_y^2 \qquad \partial_b = 0 \Rightarrow b = 0$

c) For $(x - \mathbb{E}[x]) = a(y - \mathbb{E}[y]) + b,\ b = 0$

$\Rightarrow b = \mathbb{E}[x] - a\,\mathbb{E}[y]$

d) $y = x + \varepsilon \qquad \varepsilon \perp x$

$\sigma_y^2 = \underbrace{\mathbb{E}[\varepsilon^2]}_{\sigma_n^2} + \sigma_x^2$

$\sigma_{xy}^2 = \mathbb{E}[yx] = \mathbb{E}[x(x+\varepsilon)] = \sigma_x^2$

e) $a = \dfrac{\sigma_x^2}{\sigma_x^2 + \sigma_n^2} =: r \Rightarrow \hat{x} = ry$ when $\mathbb{E}[\varepsilon] = 0$

For $b \neq 0$ & $\mathbb{E}[\varepsilon] = 0$ $\Rightarrow \hat{x} = x_0 + r(y - x_0)$ ← 18.10

$\Rightarrow \mathbb{E}[x] = \mathbb{E}[y] =: x_0$

Bernoulli

If $P_0(x) = \frac{1}{2}[\delta(x-1) + \delta(x+1)]$

$$P(y|x) \propto \exp\left[-\frac{(y-x)^2}{2\sigma_n^2}\right] \Rightarrow P(x|y) \propto \exp\left[-\frac{(y-1)^2}{2\sigma_n^2}\right]\delta_{x,1} + \exp\left[-\frac{(y+1)^2}{2\sigma_n^2}\right]\delta_{x,-1}$$

$$= \frac{e^{y/\sigma_n^2}\delta_{x,1} + e^{-y/\sigma_n^2}\delta_{x,-1}}{e^{y/\sigma_n^2} + e^{-y/\sigma_n^2}}$$

$$= \sigma(y/\sigma_n^2)\,\delta_{x,1} + \sigma(-y/\sigma_n^2)\,\delta_{x,-1}$$

$\Rightarrow \hat{x}_{MAP} = \mathrm{sgn}(y)$

$\hat{x}_{MMSE} = \tanh(y/\sigma_n^2)$ $\notin \{0,1\}$

$Var(\hat{x}_{MMSE}) < 1$

$Var(x) = 1$

Laplace

$P_0(x) = \frac{b}{2} e^{-b|x|} \qquad Var = 2/b^2$

$P(x|y) = P_0(x)\,P(y|x)$

$$\propto \exp\left[\frac{2xy - x^2}{2\sigma_n^2} - b|x|\right]$$

MMSE & MAVE estimators have ugly closed-form

posterior median

MAP is just $\hat{x} = \begin{cases} 0 & |y| < b\sigma_n^2 \\ y - b\sigma_n^2 \mathrm{sgn}(y) & \text{else} \end{cases}$

"LASSO"

## Non-Gaussian Noise

If $x$ is a centered Gausian & $\varepsilon$ is Cauchy noise

$$P(x|y) \propto \frac{e^{-x^2/2}}{(y-x)^2+1}$$

Prior

$\varepsilon, y$ don't have first moment!

$$\mathbb{E}[x|y] = y + \frac{\mathrm{Im}\,\varphi}{\mathrm{Re}\,\varphi} \qquad \varphi = e^{iy} \,\mathrm{erf}\, \frac{1+iy}{\sqrt{2}}$$

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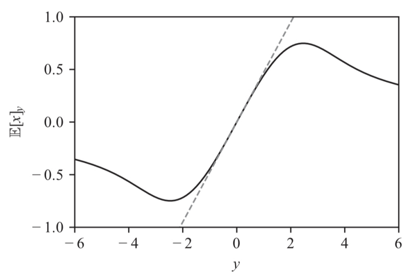



Figure 18.1 A non-monotonic optimal estimator. The MMSE estimator of a Gaussian variable corrupted by Cauchy noise (see Eq. (18.21)). For small absolute observations $y$, the estimator is almost linear with slope $2 - \sqrt{2/e\pi}/\mathrm{erfc}(1/\sqrt{2}) \approx 0.475$ (dashed line).

For small $y$, assume moderate $x, \sigma \Rightarrow$ locally linear $r \approx 1/2$

For huge $y$, must be all noise $\Rightarrow$ regress to 0

### 18.1.3 Conjugate Priors

$$P(c|y) \propto P_0(c)\, c^{-T/2} \exp\left(-\frac{y^T y}{2c}\right)$$

$$\Rightarrow P_0(c) \propto c^{-a-1} \exp(-b/c)$$

$$\Rightarrow a_p = a + T/2$$
$$b_p = b + \frac{y^T y}{2}$$

$$\Rightarrow MMSE = \mathbb{E}(c|y) = \frac{b_p}{a_p - 1} = \frac{2b + y^T y}{2(a-1) + T}$$

$$= (1-r)c_0 + r\frac{y^T y}{T} \qquad r = \frac{T}{2(a-1)+T}$$

$$T \to \infty \Rightarrow r \to 1$$

Ex 18.1.2

$$y_i \sim Lap(0, b)$$

a) $P(y|b) = \left(\frac{b}{2}\right)^T \exp\left(-b \sum |y|\right)$

b) $P(b) = Gamma(a_0, b_0) \sim b^{a_0 - 1} e^{-b_0 b}$

$$P(b|y) \propto b^T b^{a_0 - 1} e^{-b(b_0 + \sum |y|)}$$

$$a_p = a_0 + T$$
$$b_p = b_0 + \sum |y|$$

c) $MMSE = \frac{a_0 + T}{b_0 + \sum |y|}$

d) $T \to 0 \Rightarrow a_0/b_0 \qquad T \to \infty \Rightarrow m^{-1}, \quad m = \frac{1}{T}\sum |y|$

e) $MMSE = \frac{a_0 + T}{b_0 + Tm} = \frac{a_0}{b_0}(1-r) + m^{-1} r$

## 18.2 Ridge & LASSO

$$y = \sum_i a_i x_i + \varepsilon$$

$$\rightarrow y = H^T a + \varepsilon \qquad a \in \mathbb{R}^N$$
$$H \in \mathbb{R}^{N \times T}$$
$$\varepsilon \in \mathbb{R}^T$$

$$\mathbb{E}[\varepsilon \varepsilon^T] = \sigma_n^2 \mathbb{1}$$

$$\frac{1}{T} E[HH^T] = C$$

can be arbitrary
will assume $= \mathbb{1}$
(in practice how could you be sure?)
Centering won't work if $T \ll N$

$a_{reg} = (HH^T)^{-1} Hy$

When $q = N/T < 1$, $HH^T$ is invertible

$P(a|y) \propto P_0(a) \exp\left[\frac{1}{2\sigma_n^2}\|y - H^T a\|^2\right]$

Ridge: $P_0 \sim N(0, \sigma_s^2)$

$\Rightarrow P(a|y) \propto \exp\left[-\frac{1}{2\sigma_n^2}\left[a^T\left(HH^T + \frac{\sigma_n^2}{\sigma_s^2}\mathbb{1}\right)a - 2a^T Hy\right]\right]$

$\mathbb{E}[a] = \left(\frac{HH^T}{T} + \zeta \mathbb{1}\right)^{-1} \frac{Hy}{T} \qquad \zeta = \frac{\sigma_n^2}{T\sigma_s^2}$

$a_{ridge} = \underset{a}{\operatorname{argmin}} \|y - H^T a\|^2 + T\zeta \|a\|^2$

Set $\zeta$ by C.V.

LASSO

$P(a|y) \propto \exp\left[-b \sum |a_i| - \frac{1}{2\sigma_n^2}\|y - H^T a\|^2\right]$

$a_{LASSO} = \underset{a}{\operatorname{argmin}} \; 2b\sigma_n^2 |a|_1 + \|y - H^T a\|^2 \Rightarrow a_{LASSO}$ is sparse

pick by CV

## 18.2.3 In-Sample & Out-of-sample error

$a_{reg} = E^{-1} b \qquad E = \frac{1}{T} H_1 H_1^T \qquad b = \frac{1}{T} H_1^T y_1$

best in-sample estimator

Generally: $\hat{a} = \Xi^{-1} b \qquad \Xi = E$ or $\Xi = E + \zeta \mathbb{1}$

$R_{in}^2$:

$R_{in}^2 = \frac{1}{T} \mathbb{E}_{\varepsilon} \|y - H_1^T \Xi^{-1} \frac{H_1 y}{T}\|^2$

$= \frac{1}{T} \mathbb{E}_{\varepsilon} \|H_1^T a + \varepsilon - H_1^T \Xi^{-1} \frac{H_1}{T} (H_1^T a + \varepsilon)\|^2$

$$= \frac{1}{T}\Big[\sigma_\eta^2\big(\overset{\varepsilon_1^2}{T} - 2\,\mathrm{Tr}\,\Xi^{-1}E + \mathrm{Tr}\,\Xi^{-1}E\,\Xi^{-1}E\big)$$

$$+\, a^T\big(E - 2E\,\Xi^{-1}E + E\,\Xi^{-1}E\,\Xi^{-1}E\big)a\Big]$$

When $\Xi = E \in \mathbb{R}^{N\times N}$

$$= \frac{1}{T}\Big[\sigma_\eta^2(T-N) + 0\Big] = \sigma_\eta^2(1-q)$$ $\Leftarrow$ No RMT needed!

$R^2_{out}$:

$$R^2_{out} = \frac{1}{T_2}\mathop{\mathbb{E}}_{H_2,\varepsilon_2}\left\|H_2^T a + \varepsilon_2 - H_2^T\hat{a}\right\|^2$$

$$= \frac{1}{T_2}\mathop{\mathbb{E}}_{H_2,\varepsilon_2}\left\|H_2^T a + \varepsilon_2 - H_2^T\Xi^{-1}E_1 a - H_2^T\Xi^{-1}\frac{H_1\varepsilon_1}{T}\right\|^2$$

$$E_1 = T_1^{-1}H_1H_1^T$$

$$\mathop{\mathbb{E}}_{H_2} T_2^{-1}H_2H_2^T = C$$

For $\Xi = E_1$

$$\Rightarrow R^2_{out} = \frac{1}{T_2}\mathop{\mathbb{E}}_{H_2,\varepsilon_2}\left\|\varepsilon_2 - H_2^T\Xi^{-1}\frac{H_1\varepsilon_1}{T}\right\|^2 = \sigma_\eta^2 + \frac{1}{T_2}\mathop{\mathbb{E}}_{\varepsilon_1,H_2}\mathrm{Tr}\left[\underbrace{H_2H_2^T}_{T_2C}E^{-1}\frac{H_1\varepsilon_1}{T}\frac{\varepsilon_1^TH_1^T}{T}E^{-1}\right]$$

$$= \sigma_\eta^2 + \frac{\sigma_\eta^2}{T}\mathrm{Tr}\left[CE^{-1}\right]$$

$$\mathrm{Tr}\,CE^{-1} = \mathrm{Tr}\Big[C\,C^{-1/2}\underbrace{W_q^{-1}}_{\text{inv Wishart}}C^{-1/2}\Big] = \mathrm{Tr}\left[W^{-1}\right]$$

$$S_{W_q} = \frac{1}{1+qt} \Rightarrow S_{W_q^{-1}} = 1-q-qt \Rightarrow \tau(W^{-1}) = \frac{1}{S_{W_q^{-1}}(0)} = \frac{1}{1-q}$$

$$\Rightarrow R^2_{out} = \sigma_\eta^2\left(1 + \frac{N}{T}\frac{1}{1-q}\right) = \frac{\sigma_\eta^2}{1-q} = \frac{R^2_{in}}{(1-q)^3}$$

$$R^2_{in} < \sigma_\eta^2 < R^2_{out}$$

For $\Xi = E_1 + \xi \mathbb{1}$

$$R^2_{out} = \frac{1}{T_2}\mathbb{E}_{H_2,\epsilon_2} \left\| H_2^T a + \epsilon_2 - H_2^T \Xi^{-1} E_1 a - H_2^T \Xi^{-1}\frac{H_1\epsilon_1}{T}\right\|^2$$

$$= \sigma_n^2 + \mathbb{E}_{H_2}\|H_2^T \Xi^{-1}(\underbrace{\Xi - E_1}_{\xi\mathbb{1}})a\|^2 + \frac{\sigma_n^2}{T}\mathrm{Tr}(\Xi^{-1}C\Xi^{-1}E)$$

$$= \sigma_n^2 + \xi^2\, \mathrm{Tr}\, C\Xi^{-1}aa^T\Xi + \frac{\sigma_n^2}{T}\mathrm{Tr}(\Xi^{-1}C\Xi^{-1}E)$$

At $O(\xi)$ $\quad \Xi^{-1} = E^{-1} - \xi E^{-1}$

$$R^2_{out}(a_{ridge}) = R^2_{out}(a_{reg}) - \frac{2\sigma_n^2}{T}\underbrace{\mathrm{Tr}(CE^{-2})}_{\frac{N}{T}\cdot\frac{1}{N}\mathrm{Tr}\,C\,\cdot\,\frac{1}{N}\mathrm{Tr}\,E^{-2}}\xi$$

$\mathrm{Tr}\, W_q^{-2} = (1-q)^{-3}$

$$\Rightarrow R^2_{ridge} - R^2_{reg} = -\frac{2\sigma_n^2 q}{(1-q)^3}\,\tau(C^{-1})\,\xi + O(\xi^2)$$

For $\xi$ large this reverses

Note for $C = \mathbb{1}$ $\qquad \Xi = \xi\mathbb{1} + E$

$$R^2_{ridge} = \sigma_n^2 + \frac{\sigma_n^2}{T}\mathrm{Tr}(\Xi^{-1}\cancel{C}\Xi^{-1}E) + \xi^2\, \mathrm{Tr}\,\cancel{C}\Xi^{-1}aa^T\Xi^{-1}$$

$$= \sigma_n^2 + \frac{\sigma^2}{T}\mathrm{Tr}(\Xi^{-1}) - \xi\frac{\sigma_n^2}{T}\mathrm{Tr}(\underbrace{\Xi^{-2}}_{-g'_{W_q}(z)}) + \xi^2\frac{|a|^2}{N}\mathrm{Tr}(\Xi^{-2})$$

$$= \sigma_n^2(1 - q\, g_{W_q}(-\xi)) + \xi(q\sigma_n^2 - \xi|a|^2)\, g'_{W_q}(-\xi)$$

$\Rightarrow \xi_{opt}$ depends only on $|a|^2$

$$\xi = 0 \Rightarrow g_0(W_q) = -\frac{1}{1-q}$$

$$\partial_\xi R^2_{ridge} = q\sigma_n^2 g'_{W_q} + q\sigma_n^2 g'_{W_q} - 2\xi|a|^2 g'(\xi) + \xi(q\sigma_n^2 - \xi|a|^2)g''$$

$$\Rightarrow 0 = 2(q\sigma_n^2 - |a|^2\xi)\left[g'(\xi) + \frac{\xi}{2}g''(\xi)\right]$$

$$\Rightarrow \xi = q\frac{\sigma_n^2}{|a|^2} \qquad |a|^2 = \sigma_s^2$$

$$\Rightarrow R^2_{ridge} = \sigma_n^2 \left[ 1 - q\, g_{w_q}\left( -\frac{q\sigma_n^2}{|a|^2} \right) \right]$$

$-g(-\zeta)$ is monotonically decreasing $\Rightarrow$ optimal ridge is better than standard regression

## 18.3 Bayesian Estimation of C

Wishart Law (Ch 4) $\quad P(E|C) \propto (\det C)^{-T/2} \exp\left[ -\frac{T}{2} \operatorname{Tr} C^{-1} E \right]$

Conj Prior: $\quad P_0(C) = (\det C)^{a} \exp\left[ -b \operatorname{Tr} C^{-1} X \right]$

Inverse Wishart $\quad a = -\frac{T^* + N + 1}{2} \qquad b = \frac{T^* - N - 1}{2}$

$\Rightarrow \mathbb{E}_{P_0} C = X$

$$\Rightarrow P(C|E) = (\det C)^{-\frac{T + T^* + N + 1}{2}} \exp\left[ -\frac{T}{2} \operatorname{Tr}\left[ C^{-1} E^* \right] \right]$$

$$E^* = E + \frac{T^* - N - 1}{T} X$$

$$\Rightarrow \mathbb{E}[C|E] = \frac{T E^*}{T + T^* - N - 1} = rE + (1-r)X$$

$$r = \frac{T}{T + T^* - N - 1}$$

# Chapter 19 Rotationally Invariant Estimators

## 19.1 Eigenvector Overlaps

$(v_i^T u_j)^2$ Note $\sum_i (v_i^T u_j)^2 = u_j \cdot \mathbb{1} \cdot u_j = 1$

same with $\sum_j$ $\Rightarrow$ "bi-stochastic"

$v$ evecs of $E$

$$\Rightarrow G_E = \sum_j \frac{v_j v_j^T}{z-\lambda_j} \Rightarrow u^T G_E u = \sum_i \frac{(u \cdot v_i)^2}{z-\lambda_i}$$

$v_i$ continue to fluctuate as $N \to \infty$ unlike $\rho(\lambda)$

$$v_i \cdot u_j \sim O\left(\frac{\sigma_{ij}}{\sqrt{N}}\right) \Rightarrow N \cdot (v_i \cdot u_j)^2 \sim O(\sigma_{ij})$$

$$\Phi(\lambda_i, \mu_j) := N\, \mathbb{E}\, (v_i \cdot u_j)^2$$

$\mathbb{E}$ is either over realizations of $E$

or otherwise avg over interval of width $d\lambda = \eta$

$N^{-1} \ll \eta \ll 1$ eg $N^{-1/2}$

These are the same by self-averaging property

$$\operatorname{Im} u_j^T G(\lambda_i - i\eta)\, u_j = \pi \rho_E(\lambda_i)\, \Phi(\lambda_i, \mu_j)$$

For $E, E'$ (with same population cov $C$)

$$\Psi(\lambda_i, \lambda_j') := N\, \mathbb{E}[(v_i \cdot v_j')^2]$$

Consider $\psi := \frac{1}{N} \operatorname{Tr} G_E(z) G_{E'}(z') = \frac{1}{N^2} \sum_{i,j} \frac{N (v_i^T v_j')^2}{(z-\lambda_i)(z'-\lambda_j')}$

$$\Rightarrow \operatorname{Re}\left[\psi(\lambda_i - i\eta, \lambda_j + i\eta) - \psi(\lambda_i - i\eta, \lambda_j' - i\eta)\right] = 2\pi^2 \rho_E(\lambda_i) \rho_{E'}(\lambda_j')\, \Psi(\lambda_i, \lambda_j') \;*$$

$$\left[i\pi\rho(\lambda_i)\cdot(-i\pi\rho(\lambda_j)) - i\pi\rho(\lambda_i)\, i\pi\rho(\lambda_j)\right] \Psi(\lambda_i, \lambda_j)$$

### 19.1.2 Overlaps in the Additive Case

From replica calc'n:

$$\mathbb{E}\, G_E(z) = G_C(z - R_X(g_E(z)))$$

$$u_j^T G_E(\lambda_i - i\eta) u_j = \frac{1}{\lambda_i - i\eta - R_X(g_E(\lambda_i - i\eta)) - \mu_j}$$

evec of C

as $\eta \to 0$ Im gives $\Phi(\lambda_i, \mu_j)$

For $R_X(z) = \sigma^2 z$:

$$u_j^T G_E(\lambda_i - i\eta) u_j = \frac{1}{\lambda_i - i\eta - \sigma^2 \underbrace{g_E}_{g_E := h_E + i\pi\rho_E}(\lambda_i - i\eta) - \mu_j} = \frac{1}{\lambda_i - \mu_j - \sigma^2 h_E - i\pi\sigma^2\rho_E - i\eta}$$

$$= \frac{i\pi\sigma^2\rho_E + i\eta}{(\mu_j - \lambda_i + \sigma^2 h_E)^2 + \pi^2\sigma^4\rho^2}$$

$$\Rightarrow \Phi(\mu, \lambda) = \frac{\sigma^2}{(\mu - \lambda + \sigma^2 h_E(\lambda))^2 + \sigma^4\pi^2\rho_E(\lambda)^2}$$

$\Rightarrow$ Overlap peaks for $\mu = \lambda - \sigma^2 h_E(\lambda)$

$\sigma^2 \to 0 \Rightarrow \Phi(\mu, \lambda) = \delta(\mu - \lambda)$ ie evecs are equal

$\Rightarrow$ Overlaps of $u_i \cdot v_j$ are $\sim N^{-1/2}$ for $\sigma > 0$

Let $E = C + X \quad E' = C + X' \quad X \perp X'$

$*\ \Psi(z, z') = \frac{1}{N} \mathrm{Tr}\, G_E(z) G_{E'}(z') \qquad \mathbb{E}\, G_X = \mathbb{1} g_X$

$$G_E(z) = (z - E)^{-1} = (\underbrace{z - \sigma^2 g_E}_{=\zeta} - C)^{-1}$$

$$\Rightarrow \Psi = \mathrm{Tr}\left[(z-E)^{-1}(z'-E')^{-1}\right]$$

1603.04364

FINISH

$$= \mathrm{Tr}\left[(\zeta - C)^{-1}(\tilde{\zeta} - C)^{-1}\right]$$

Key step

$$= \frac{1}{\zeta' - \zeta} \mathrm{Tr}\left[(\zeta - C)^{-1} - (\zeta' - C)^{-1}\right]$$

$$= -\frac{g_E(z) - g_E(z')}{\zeta(z) - \zeta(z')} \qquad \text{for } \Psi(\zeta, \zeta)$$

$$\Psi(\lambda - i\eta, \lambda + i\eta) - \Psi(\lambda - i\eta, \lambda - i\eta)$$

$$= \frac{g(\zeta' - \bar{\zeta}') + \zeta(\bar{g}' - g') + g\bar{\zeta}' - \bar{g}'\zeta'}{(\zeta - \zeta')(\zeta - \bar{\zeta}')} \to \frac{2\mathrm{Re}\left[\sigma^2(\lambda - \lambda')(\zeta - \zeta')\right]}{(\zeta - \zeta')^2} = \frac{2\sigma^2}{\partial_\lambda \zeta}$$

$$\Rightarrow \Psi(\lambda\,\lambda) = \frac{1}{2\pi^2 \rho_E(\lambda)^2}\, Re\left[\frac{1}{\partial_\lambda(\lambda - \sigma^2 g_E)}\right]$$

## 19.1.3 Overlaps in Multiplicative case

$$E = C^{1/2} W_q C$$

$$G_E = \frac{Z(z)}{z} G_C(Z(z)) \qquad Z = \frac{z}{1-q+qz\,g_E(z)}$$

$$u_j^T G_E(\lambda_i - i\eta) u_j = \frac{Z(\lambda_i - i\eta)}{\lambda_i - i\eta} \frac{1}{Z(\lambda_i - i\eta) - \mu_j}$$

$$= \frac{1}{\mu(q-1) - g_E q \lambda\mu + \lambda - i\eta} = \frac{q\mu\lambda\,\pi\rho_E}{(\mu(1-q) - \lambda + q\mu\lambda h_E)^2 + q^2\mu^2\lambda^2\pi^2\rho_E^2}$$

$q \to 0 \Rightarrow$ sharp peak near $\lambda = \mu$

Now $E = C^{1/2} W_q C^{1/2}$ $\quad E' = C^{1/2} W_q' C^{1/2}$

→ Annoying computation 1603.04364

For $C \to \mathbb{1}$, $\tau(C^2) = 1 + \epsilon$

$$\Rightarrow \Psi(\lambda, \lambda') = 1 + \epsilon[2h_E(\lambda) - 1][2h_E(\lambda') - 1] + O(\epsilon^2)$$

Overlaps again depend only on $g_E$ $\Rightarrow$ hypothesis that $E, E'$ came from same $C$ only needs $\lambda_E, \lambda'_E$

## 19.2 Rotationally Invariant Estimators

Our priors are rotationally invariant

$$P_0(C) = P_0(OCO^T)$$

$$P(C|E) \propto \det C^{-T/2} \exp\left[-\tfrac{T}{2} \mathrm{Tr}\, C^{-1} E\right] P_0(C)$$

$$\mathbb{E}(C \mid OEO^T) = O\, \mathbb{E}(C|E)\, O^T$$

More generally $\Xi(C)$ is a RIE if

$$\Xi(OEO^T) = O\,\Xi(E)\,O^T$$

$\Xi(E)$ can be diagonalized in the same basis as $E$ up to a fixed rotation $\Omega$

There is no natural guess for $\Omega$ except $\Omega = \mathbb{1}$

$$\Rightarrow \Xi(E) = \sum_{i=1}^{N} \xi_i v_i v_i^T \leftarrow \text{evecs of } E$$

function of empirical $\lambda_i$

Goal is to choose $\xi$ optimally so that $\Xi(E)$ is as close to $C$ as possible

If $\vec{v}_E = \vec{v}_C$ then $\xi_i = \mu_i$

Generally want to minimize:

$$\mathrm{Tr}\left[(\Xi(E) - C)^2\right] = \sum_i v_i^T (\Xi(E) - C)^2 v_i$$
$$= \sum_i \xi_i^2 - 2\xi_i v_i^T C v_i - v_i^T C^2 v_i$$

$$\frac{\partial}{\partial \xi_i} = 0 \Rightarrow \xi_i = v_i^T C v_i \qquad \text{"Oracle estimator"}$$

Can't compute $\xi$ without knowing $C$ it seems!

## 19.2.3 The Large Dimension Miracle

$$\xi_k = \sum_j v_k^T u_j \mu_j u_j^T v_k$$
$$= \sum_j \mu_j (v_k^T u_j)^2 \rightarrow \int d\mu\, \rho_C(\mu)\, \Phi(\lambda_k, \mu)$$
$$= \frac{1}{\pi \rho_E(\lambda_k)} \lim_{\eta \to 0^+} \mathrm{Im} \sum_j \mu_j u_j^T G_E(\lambda_k - i\eta) u_j$$
$$= \frac{1}{\pi \rho_E(\lambda_k)} \lim_{\eta \to 0^+} \mathrm{Im}\, \mathrm{Tr}\left[C\, G_E(\lambda_k - i\eta)\right]$$

In both the additive & multiplicative case:

$$G_E(z) = Y(z)\, G_C(Z(z)) \quad \} \text{ Subordination}$$

$Y(z) = 1$ for $+$

$Y(z) = \frac{Z(z)}{z}$ for $\times$

$$\Rightarrow \tau[C G_E] = Y(z)\, \tau[C G_C(Z)]$$
$$= Y(z) Z(z) g_C(Z) - Y(z)$$
$$= Z(z) g_E(z) - Y(z)$$

But $Z(z)$ depends only on $g_E$

$\Rightarrow \tau[C G_E]$ doesnt depend on $g_C$ !

$\xi_k = \frac{1}{\pi \rho_E(z)} \lim_{\eta \to 0^+} \operatorname{Im} Z(z_k) g_E(z_k) - Y(z_k)$ } estimatable from data alone!

## 19.2.4 Additive Case:

$Z(z) = z - R_X(g_E(z))$ $\quad Y = 1$

$$\xi(\lambda) = \lambda - \frac{\lim_{\eta \to 0} \operatorname{Im} R_X(g_E(z)) g_E(z)}{\lim_{\eta \to 0} \operatorname{Im} g_E}$$

$X = 0 \Rightarrow R = 0 \Rightarrow \xi = \lambda$ as expected

$X$ small $\Rightarrow R = \varepsilon x + \cdots$

$\tau(X^2)$ $\quad \tau(X) = 0$

$$= \lambda - \frac{2\epsilon h_E(\lambda) \pi \rho_E(\lambda)}{\pi \rho_E} = \lambda - 2\epsilon h_E(\lambda)$$

Exact for Wigner noise $R_X(\lambda) = \sigma_n^2 \lambda$

If $C$ is also Wigner with $\sigma_s^2 \Rightarrow E$ is Wigner with $\sigma_s^2 + \sigma_n^2$

$h_E(z) = \frac{z}{2\sigma^2}$ for Wigner $\quad \sigma^2 = \sigma_s^2 + \sigma_n^2$

$$\Rightarrow \xi(\lambda) = \lambda\left(1 - \frac{\sigma_n^2}{\sigma^2}\right) = \lambda \underbrace{\left(\frac{\sigma_s^2}{\sigma^2}\right)}_{:= r}$$

linear shrinkage from last chapter! $(x_0 = 0)$

$\Rightarrow \Xi(E)_{ij} = r E_{ij}$

Ex 19.2.1

$E = C + X$ $\quad$ $C, X$ from same dist mutually free

a) $R_X(g) + R_C(g) = R_E(g)$

$$\Rightarrow 2R_X(g) = R_E(g)$$

b) $g_E R_E(g) = z g_E - 1$

$$\Rightarrow g_E R_X(g) = \frac{1}{2}(z g_E - 1)$$

c) $\Rightarrow \xi(\lambda) = \lambda - \frac{1}{2} \frac{\lim_{\eta \to 0^+} \operatorname{Im} z g_E}{\lim_{\eta \to 0^+} \operatorname{Im} g_E} = \frac{\lambda}{2}$

d) $\Xi = \mathbb{E}[C|E] \Rightarrow \Xi = E/2$

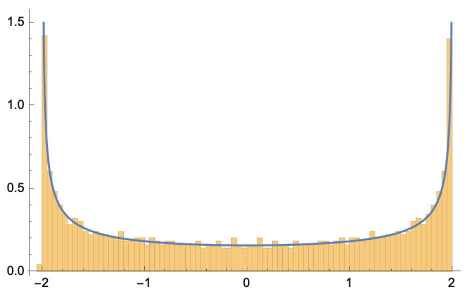
e)

In[48]:=
```
n = 1000;
X1 = RandomVariate[GaussianOrthogonalMatrixDistribution[1, n]] / Sqrt[n / 2];
X2 = RandomVariate[GaussianOrthogonalMatrixDistribution[1, n]] / Sqrt[n / 2];
{vals, vecs} = Eigensystem[X1];
sX1 = Transpose[vecs].DiagonalMatrix[Sign[vals]].vecs;
{vals, vecs} = Eigensystem[X2];
sX2 = Transpose[vecs].DiagonalMatrix[Sign[vals]].vecs;
{vals, vecs} = Eigensystem[sX1 + sX2];
```

f)

In[59]:=
```
Show[Histogram[vals, 100, "PDF"], Plot[1/(π Sqrt[4 - λ^2]), {λ, -2, 2}, PlotRange → 1.5]]
```

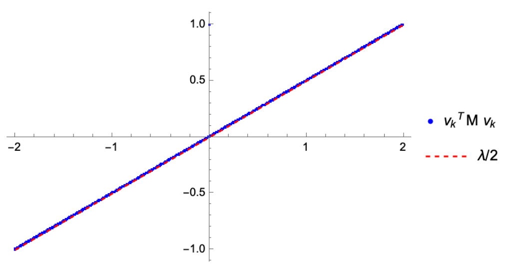
Out[59]=




g)

In[72]:=
```
Show[ListPlot[Transpose[{vals, Diagonal[vecs.sX1.Transpose[vecs]]}],
  PlotStyle → Blue, PlotLegends → {"v_k^T M v_k"}],
 Plot[λ / 2, {λ, -2, 2}, PlotStyle → {Red, Dashed}, PlotLegends → {"λ/2"}]]
```

Out[72]=



## 19.2.5 Multiplicative Case

$$\xi(\lambda) = \frac{1}{\pi \rho_E(\lambda_k)} \lim_{\eta \to 0^+} \operatorname{Im} \tau\left[C\, G_E(\lambda_k - i\eta)\right] = \lim_{\eta \to 0^+} \operatorname{Im} \frac{1}{\lambda_k - i\eta} \frac{\tau\left[C\, T_E(\lambda_k - i\eta) + C\right]}{\pi \rho_E(\lambda_k)}$$

$$= \frac{\lim_{\eta \to 0^+} \operatorname{Im} \tau\left[C\, T_E(z)\right]}{\lim_{\eta \to 0^+} \operatorname{Im} t_E(z)}$$

$$T_E(z) = T_C\left[z\, S_W(t_E(z))\right]$$

$$\Rightarrow \xi(\lambda) = \frac{\lim_{\eta \to 0^+} \operatorname{Im} \tau\left[C\, T_C(z\, S_W(t_E(z)))\right]}{\lim_{\eta \to 0^+} \operatorname{Im} t_E(z)}$$

$$= \tau\left[C^2 (z S_W(t_E) - C)^{-1}\right]$$

$$= -\tau[C] + z S_W(t_E)\, t_C(z S_W(t_E))$$

$$= -\underbrace{\tau[C]}_{\text{real}} + z S_W(t_E)\, t_E(z)$$

$$\Rightarrow \xi(\lambda) = \lambda \lim_{\eta \to 0^+} \frac{\operatorname{Im} S_W(t_E(z))\, t_E(z)}{\operatorname{Im} t_E(z)} \qquad z = \lambda - i\eta \qquad \text{general}$$

$$E = C^{1/2} W_q C^{1/2} \Rightarrow S_{W_q} = (1+qt)^{-1}$$

Im $tF = 0$

$$\xi(\lambda) = \lambda \lim_{\eta \to 0^+} \frac{\operatorname{Im} \frac{t}{1+qt}}{\operatorname{Im} t} = \frac{\lambda}{|1+qt_E|^2} \frac{\operatorname{Im} t_E(1+qF)}{\operatorname{Im} t_E}$$

nonlinear "shrinkage" $\rightarrow$

$$= \left.\frac{\lambda}{|1+q t_E(\lambda - i\eta)|^2}\right|_{\eta \to 0^+}$$

$$t_E = \lambda g_E(\lambda) - 1 \qquad = \left.\frac{\lambda}{|1 - q + q g_E(\lambda - i\eta)|^2}\right|_{\eta \to 0^+}$$

$\lambda > 0$ for cov $\Rightarrow t(\lambda_-) < 0$

$t(\lambda_+) > 0$

For $C = M_p$ inverse Wishart

$$\Rightarrow t_E = \frac{z-q-1 \pm \sqrt{(z-q-1)^2 - 4(q+zp)}}{2(q+zp)} \qquad \text{from } 15.4$$

$$\Rightarrow \xi(\lambda) = \frac{q+\lambda p}{p+q} = r\lambda + (1-r) \qquad r := \frac{p}{p+q}$$

Bayesian shrinkage as in 18.3

for $\lambda \in [\lambda_-, \lambda_+]$

for $\lambda$ outside this, $\xi$ is nonlin in $\lambda$

## Ex 19.2.2 RIE for C Wishart

$$C = W_{q_0}$$

$$E = C^{1/2} W_q C^{1/2}$$

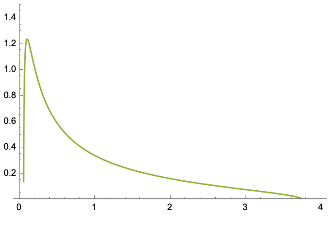
a) $$S_E = \frac{1}{(1+q_0 t)(1+qt)}$$

$$\Rightarrow t\xi_E = (1+q_0 t)(1+qt)(1+t) \quad \xi \text{ Cubic}$$

b)

```
In[212]:= solns = Solve[t ξ == (1 + q0 t) (1 + q t) (1 + t), t] /. q0 → 1 / 4 /. q → 1 / 2;
Plot[1/π Im[(t + 1)/ξ] /. solns // Evaluate, {ξ, 0, 4}, PlotPoints → 400, PlotRange → {0, 1.5}]
```

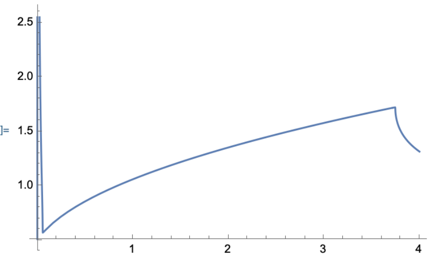
Out[213]=




c)

```
In[243]:= Plot[λ / (Abs[1 + q t]^2 /. q → 1 / 2 /. solns[[2]] /. ξ → λ - I 0.0001), {λ, 0, 4}]
```

Out[243]=



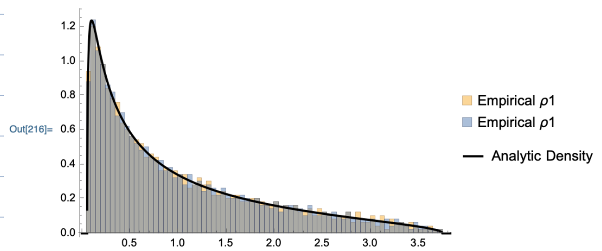
d)

```
In[172]:= n = 1000;
q0 = 1 / 4;
q = 1 / 2;
t0 = n / q0;
t1 = n / q;
C0 = RandomVariate[WishartMatrixDistribution[t0, IdentityMatrix[n] / t0]];
W1 = RandomVariate[WishartMatrixDistribution[t1, IdentityMatrix[n] / t1]];
W2 = RandomVariate[WishartMatrixDistribution[t1, IdentityMatrix[n] / t1]];
SqrtC = MatrixPower[C0, 1 / 2];
E1 = SqrtC.W1.SqrtC;
E2 = SqrtC.W2.SqrtC;

In[183]:= τ[M_] := Tr[M] / Length[M]
{τ[C0], τ[W1], τ[W2], τ[E1], τ[E2]}
{τ[C0.C0], τ[W1.W1], τ[W2.W2], τ[E1.E1], τ[E2.E2]}

Out[184]= {0.999531, 0.998954, 1.00131, 0.998242, 1.00131}

Out[185]= {1.24894, 1.49946, 1.50357, 1.74519, 1.75685}
```

e)

```
In[216]:= Show[Histogram[{Eigenvalues[E1], Eigenvalues[E2]}, 100, "PDF",
   ChartLegends → {"Empirical ρ1", "Empirical ρ1"}],
  Plot[1/π Im[(t - 1)/ζ] /. solns[[3]] // Evaluate, {ζ, 0, 4}, PlotPoints → 400,
   PlotStyle → {Black, Thick}, PlotLegends → {"Analytic Density"}]]
```

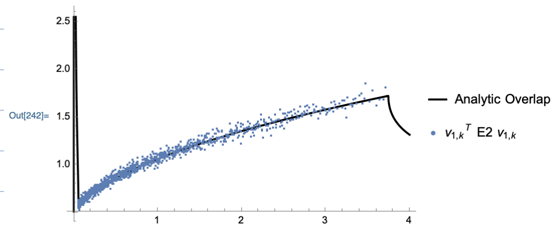
Out[216]=




f)

```
In[240]:= {evals1, evecs1} = Eigensystem[E1];
overlaps = Diagonal[evecs1.E2.Transpose[evecs1]];
Show[Plot[λ / (Abs[1 + q t]^2 /. q → 1 / 2 /. solns[[2]] /. ζ → λ - I 0.0001), {λ, 0, 4},
   PlotStyle → {Thick, Black}, PlotLegends → {"Analytic Overlap"}],
  ListPlot[Transpose[{evals1, overlaps}], PlotLegends → {"v1,k^T E2 v1,k"}]]
```

Out[242]=




Ex 19.2.3

a) For W, C drawn from same ensemble

$$S_E = S_C^2 = S_W^2 \Rightarrow S_W = \sqrt{S_E}$$

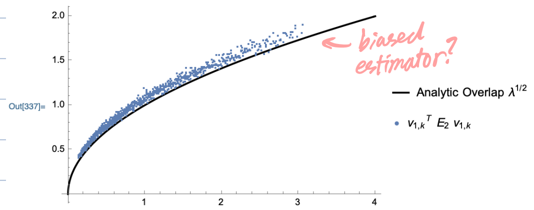
$$\Rightarrow t_E\sqrt{S_E(t_E(z))} = \sqrt{\frac{t_E+1}{t_E z}}\cdot t_E \Rightarrow \sqrt{\lambda}\ \frac{\mathrm{Im}\sqrt{t_E^2+t_E}}{\mathrm{Im}\ t_E}$$

$$= \sqrt{\lambda}\ \frac{\mathrm{Im}\ t_E\sqrt{1+t_E^{-1}}}{\mathrm{Im}\ t_E}$$ ← real?

b)

```
In[335]:= {evals1, evecs1} = Eigensystem[E1];
overlaps = Diagonal[evecs1.E2.Transpose[evecs1]];
Show[Plot[Sqrt[λ], {λ, 0, 4}, PlotStyle → {Thick, Black},
  PlotLegends → {"Analytic Overlap λ^1/2"}],
 ListPlot[Transpose[{evals1, overlaps}], PlotLegends → {"v1,k^T E2 v1,k"}]]
```

Out[337]=




## 19.2.6 RIE for outliers

Assume $C$ has outliers that appear as outliers of $E$

For $z$ outside the bulk $g_E$, $t_E$ analytic

$$\Rightarrow \operatorname{Im} g_E(\lambda - i\eta) = -\eta\, g_E'(\lambda)$$

$$\operatorname{Im} t_E(\lambda - i\eta) = -\eta\, t_E'(\lambda)$$

Additive Case:

$$\Rightarrow \xi(\lambda) = \lambda - \frac{(R(g)g)'}{g'} = \lambda - R(g) - R'(g)\,g \qquad ' = \partial_z$$

$$= \lambda - \frac{d}{dg}\left[g R_x(g)\right]$$

Multiplicative Case:

$$\Rightarrow \xi(\lambda) = \lambda \frac{d}{dt}\left[t\, S_w(t)\right]$$

## 19.3 Properties of optimal RIE for Covariance Matrices

What is the effect of $\lambda_i \to \xi(\lambda_i)$

$$\operatorname{Tr} \Xi = \sum_j \mu_j\, u_j^T \Big(\sum_i v_i v_i^T\Big) u_j = \operatorname{Tr} C$$

$\Rightarrow$ Cleaning preserves trace

$$\mathrm{Tr}\,\Xi = \sum_{jk} \mu_j \mu_k \underbrace{\sum_i (u_j \cdot v_i)^2 (u_k \cdot v_i)^2}_{=: A_{jk}}$$

$$\sum_j A_{jk} = \sum_{j,i} v_i^T u_j u_j^T v_i \, v_i^T u_k u_k^T v_i$$

$$= \sum_i v_i^T u_k u_k^T v_i = \mathrm{Tr}\, u_k u_k^T = 1$$

Similarly for $\sum_k A_{jk} = 1$

$$\Rightarrow \sum_{j,k} A_{jk} \mu_j \mu_k \le \sum_j \mu_j^2$$

$$\Rightarrow \mathrm{Tr}\,\Xi^2 \le \mathrm{Tr}\, C^2 \le \mathrm{Tr}\, E^2$$

Analogue of optimal ridge?

Be more cautious than just "bringing back" the sample $\lambda_i$

Always shrink top $\lambda$ down / bottom $\lambda$ up as in $\lambda/2$ for additive case, $\sqrt{\lambda}$ for mult. case

Asymptotic behavior:

Assume outlier to the left of lower bound $\rho_E$ & $q<1$ $\Rightarrow$ E has no 0 modes

$\lambda g_E(\lambda)$ is $O(\lambda)$ for $\lambda \to 0$

$$\Rightarrow 1+q\, t_E(\lambda) = 1-q + O(\lambda)$$

$$\Rightarrow \xi(\lambda) = \frac{\lambda}{(1-q)^2} + O(\lambda^2)$$

small $\lambda$ grow ✓

Now assume $\lambda \to \infty$

$$\lim_{\lambda\to\infty} \lambda t_E \sim \frac{\tau(E)}{\lambda} \Rightarrow \xi = \frac{\lambda}{\left|1 + q\frac{\tau(E)}{\lambda} + O(\lambda^{-2})\right|^2} \approx \lambda - 2q\tau(E) + O(\lambda^{-2})$$

$$\tau(E) = 1 \Rightarrow \xi = \lambda - 2q + O(\lambda^{-1})$$

From 19.54 a+1 outlier eig will be shifted by q

$$\lambda_1 = \underbrace{a+1}_{\mu} + q$$

$\Rightarrow \xi \approx \mu - q$

$\Rightarrow \xi < \mu < \lambda$

## 19.4 Conditional Average in Free Probability

Alternative derivation of

$$\xi_k = \frac{1}{\pi \rho_E(\lambda_k)} \lim_{\eta \to 0^+} \operatorname{Im} \tau[C G_E(\lambda_k - i\eta)]$$

E obtained by free operations on C

Best MSE estimator is $\Xi(E) = \mathbb{E}[C|E]$

$\Xi$ is a function of E only (since only E is known)

$\Rightarrow [\Xi, E] = 0$

$\Rightarrow$ Compute $m_k = \tau[\Xi E^k]$

Generating function: $F(z) = \tau[\xi(E)(z-E)^{-1}]$
at $z \to \infty$

$$= \tau[\mathbb{E}[C|E](z-E)^{-1}]$$

But $\tau$ contains $\mathbb{E}$

$\Rightarrow \tau(\mathbb{E}[\cdot]) = \tau(\cdot)$ $= \tau[C(z-E)^{-1}]$ $= \int \rho_E(\lambda) \frac{\xi(\lambda)}{z-\lambda} d\lambda$

$$\Rightarrow \lim_{\eta \to 0^+} \operatorname{Im} F(\lambda - i\eta) = \pi \rho_E \xi(\lambda)$$

$$\Rightarrow \xi(\lambda) = \lim_{\eta \to 0^+} \frac{\operatorname{Im} F(\lambda - i\eta)}{\operatorname{Im} g_E(\lambda - i\eta)}$$

## 19.5 Real Data

### 19.5.1 Parametric Approach

Postulate a form for $\rho_C$

Simplest is $\rho_C \sim$ Inverse Wishart

$\Rightarrow E = \sqrt{M_p}\, W_q \sqrt{M_p} \Rightarrow \rho_E = \dfrac{\sqrt{4(p\lambda+q)-(1+q-\lambda)^2}}{2\pi\lambda q p}$

$\tau(E^2) = 1+p+q$

$\tau(E^{-1}) = \dfrac{1+p}{1-q}$ $\{ \tau(W_q^{-1}M_p^{-1}) = \tau(W_q^{-1})\tau(M_p^{-1})$

When the empirical $\rho$ seems to be bounded above and below
a more general ansatz:

$$S_C = \frac{(1-p_1 t)(1-p_2 t)}{(1+q_1 t)} \Leftrightarrow S_E(t) = \frac{(1-p_1 t)(1-p_2 t)}{(1+q t)(1+q_1 t)}$$

$\rightsquigarrow$ cubic for $t$ (and hence $\rho$)

$p_1, p_2, q_1, \ldots$ fitted from moments of $E, E^{-1}$ (equiv. from density)

Doesnt work super well...

Alternatively, postulate a form eg

$$\rho_E = Z^{-1} \frac{(1+a_1\lambda + a_2\lambda^2)\sqrt{(\lambda-\lambda_-)(\lambda_+-\lambda)}}{1+b_1\lambda+b_2\lambda^2}$$

Through eg MSE on CDF

$\Rightarrow$ Then reconstruct $g_E(x-i0^+)$ numerically
from fitted $\rho$ + its Hilbert transform $⨍ \frac{\rho(x)}{z-\lambda} d\lambda$

But even when $\rho$ fits the sample density,
it cannot be obtained as a free prod of Wishart w/ some population density

$\Rightarrow$ Approximate estimator is non-monotonic in true estimator
Believed that this should never be the case

For unbounded support (eg sharp left edge but unbounded right tail)

$$\rho_C = \Theta(\lambda - \lambda_-) \frac{2}{a\sqrt{\pi\mu}} \frac{\Gamma(\frac{1+\mu}{2})}{\Gamma(\frac{\mu}{2})} \left(1 + \frac{(\lambda-\lambda_-)^2}{a^2\mu}\right)^{-\frac{1+\mu}{2}}$$

$\Rightarrow \rho(\lambda) \sim \lambda^{-\mu-1}$ as $\lambda \to \infty$

## 19.5.2 Kernel Methods

$$\rho_s(\lambda) = \frac{1}{N} \sum_{k=1}^{N} K_{\eta_k}(\lambda - \lambda_k)$$

possibly $k$-dep width $\eta_k$

$$\int du\, K_\eta(u) = 1 \Rightarrow \int dx\, \rho_s(x) = 1$$

$$\Rightarrow g_s(z) := \frac{1}{N} \sum_{k=1}^{N} g_{K;\eta_k}(z - \lambda_k)$$

$$g_{K\eta_k} = \int_{-\infty}^{\infty} du \frac{K_{\eta_k}(u)}{z-u} \qquad \operatorname{Im}(z) \neq 0$$

$$\operatorname{Im} g_{K_\eta}(x - i0^+) = i\pi K_\eta(x)$$

$$\Rightarrow \operatorname{Im} g_s(x - i0^+) = i\pi \rho_s(x) \qquad \forall K_\eta$$

$$\Rightarrow h_s(x) := \operatorname{Re} g_s(x) = ⨍ d\lambda \frac{\rho_s(\lambda)}{z-\lambda}$$

Particularly relevant kernels are:

Cauchy: $K_\eta^C(x) = \frac{1}{\pi} \frac{\eta}{u^2+\eta^2}$

$$\Rightarrow g_{K^C;\eta}(z) = \frac{1}{z \pm i\eta} \Rightarrow g_{s,\eta}^c(z) = \frac{1}{N} \sum_k \frac{1}{z - \lambda_k - i\eta_k} \qquad \operatorname{Im}(z) < 0$$

Another choice: Semicircle "Wigner" Kernel

$$K_\eta^W(u) = \frac{\sqrt{4\eta^2 - u^2}}{2\pi\eta^2}$$

$$g_s^W(z) = \frac{1}{N} \sum_{k=1}^{N} \frac{z-\lambda_k}{2\eta_k^2} \left[1 - \sqrt{1 - \frac{4\eta^2}{(z-\lambda_k)^2}}\right]$$

1. Can rectify nonmonotonicity of $\xi(\lambda)$ by hand

2. Usually want $\xi(\lambda)$ exactly at $\lambda = \lambda_k$

empirically, excluding $\lambda_k$ from kernel estimator gives consistently better results

## 19.6 Validation & RIE

$$\xi_X(\lambda_i) := v_i E' v_i$$

$v_i$ computed from train set $E$

$E'$, $\xi_X$ "out of sample"

If $C$ is the same $E = \sqrt{C} W \sqrt{C}$ $E' = \sqrt{C} W' \sqrt{C}$ $W' \perp W$

$$\Rightarrow \xi_X = \sum_{k=1}^{N} (v_i^T v_k')^2 \lambda_k' \rightarrow \int d\lambda' \rho_{E'}(\lambda') \lambda' \Psi(\lambda, \lambda')$$

$$\Psi(\lambda, \lambda') = \frac{1}{N} \sum_j \underbrace{\Phi(\lambda, \mu_j)}_{E, C} \underbrace{\Phi(\lambda', \mu_j)}_{E', C} \Big\} \text{ exact}$$

$$\rightarrow \int \rho_C(\mu) \Phi(\lambda, \mu) \Phi(\lambda', \mu) \, d\mu$$

Proof:

$$v_i^E = \frac{1}{\sqrt{N}} \sum_j \varepsilon_{ij} \sqrt{\Phi(\lambda_i, \mu_j)} \, u_j \qquad v_i^{E'} = \frac{1}{\sqrt{N}} \sum_j \varepsilon_{ij}' \sqrt{\Phi(\lambda_i', \mu_j)} \, u_j$$

$$\mathbb{E}[\varepsilon \varepsilon'] = 0 \qquad \mathbb{E}[\varepsilon] = \mathbb{E}[\varepsilon'] = 0 \qquad \mathbb{E}[\varepsilon_{ij} \varepsilon_{kl}] = \mathbb{E}[\varepsilon_{ij}' \varepsilon_{kl}'] = \delta_{ik} \delta_{jl}$$

"Ergodic assumption" (Justifiable by DBM of evecs)

$$\Rightarrow \mathbb{E}[(v_i^T v_k')^2] = \frac{1}{N} \sum_j \Phi(\lambda_i, u_j) \Phi(\lambda_k, u_j)$$

$$\Rightarrow \xi_X(\lambda) = \frac{1}{N^2} \sum_{k,j} \Phi(\lambda, \mu_j) \Phi(\lambda_k', \mu_j) \lambda_k'$$

$$= \frac{1}{N^2} \sum_j \Phi(\lambda, \mu_j) \sum_k \Phi(\lambda_k', \mu_j) \lambda_k'$$

$= N u_j^T E' u_j$

$= N u_j^T \sqrt{C}\, W' \sqrt{C}\, u_j$

$= N \mu_j \; u_j^T W' u_j$ ← since $u$ is eig of $W$

Since $C \perp W'$ $\underset{[\text{internal}]}{\int} d\mu \, \rho(\mu) \, [\cdots]$

involves $W'$ wedged between randomly oriented evecs

$\mathbb{E}[u^T W u] = \tau(W')\tau(uu^T) = 1$

$\Rightarrow \xi_\times(\lambda) = \frac{1}{N} \sum_j \Phi(\lambda, \mu_j)\mu_j \rightarrow \int \rho_C(\mu)\Phi(\lambda,\mu)\, d\mu = \xi(\lambda)$

$\Rightarrow$ Can approximate $\xi$ by considering $v_i^T E' v_i$ even when $E, E'$ have different $q$



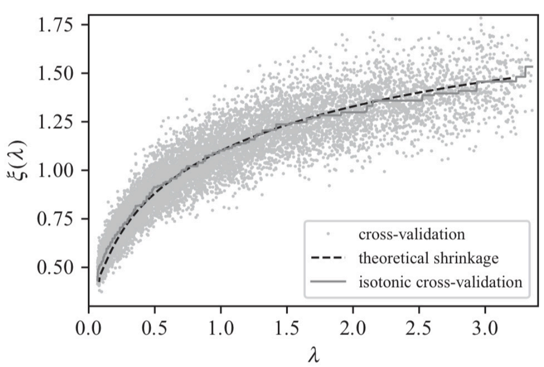



Figure 19.5 Shrinkage function $\xi(\lambda)$ computed for the same problem as in Figure 19.3, now using cross-validation. The dataset is divided into $K = 10$ blocks of equal length. For each block, we compute the $N = 1000$ eigenvalues $\lambda_i^b$ and eigenvectors $\mathbf{v}_i^b$ of the sample covariance matrix using the rest of the data (of new length $9T/10$), and compute $\xi_\times(\lambda_i^b) := \mathbf{v}_i^{b\,T}\mathbf{E}'\mathbf{v}_i^b$, with $\mathbf{E}'$ the sample covariance matrix of the considered block. The dots correspond to the $10 \times 1000$ pairs $(\lambda_i^b, \xi_\times(\lambda_i^b))$. The full line is an isotonic regression through the dots. The procedure has a slight bias as we in fact compute the optimal shrinkage for a value of $q$ equal to $q_\times = 10N/9T$, but otherwise the agreement with the optimal curve is quite good.

## Chapter 20: Applications to Finance

$N$ assets at time $t$ with prices $p_{it}$

Returns: $r_{it} : \dfrac{p_{it} - p_{i,t-1}}{p_{i,t-1}}$

Total capital: $C$

risk free rate

Return at $t$: $R_t = \sum_j \pi_j r_{jt} + (C - \sum_j \pi_j) r_0$

$\Rightarrow$ Excess return $= R_t - C r_0$

$= \sum \pi_i (r_{it} - r_0)$

From now on, denote $r_{it} - r_0$ by $r_{it}$

assume $\vec{r}_t \sim \vec{g}$ "gains"

$C_{ij} = Cov[r_i r_j]$

assume gains are known

Not wise to swap $E$ for $C$

$\Rightarrow$ evals can differ meaningfully

Limited data $\Rightarrow T$ is not huge

### 20.1.2 Portfolio Risk

$$R^2 := Var R = \sum_{ij} \pi_i \pi_j Cov[r_i r_j] = \pi^T C \pi$$

Expected shortfall at $p$th quantile

$$S_p = -\frac{1}{p} \int_{-\infty}^{R_p} dR \, R \, P(R)$$

$R_p$ s.t. $\int_{-\infty}^{R_p} dR \, P(R) = p$

## 20.1.3 Markowitz Portfolio theory

$$\min \quad \pi^T C \pi$$
$$\text{s.t.} \quad \pi^T g \geq G$$

expected return target

$$\Rightarrow \min_{\pi} \; \pi^T C \pi - \gamma \pi^T g$$

$$\Rightarrow \pi_C = G \frac{C^{-1} g}{g^T C^{-1} g}$$

} Needs knowledge of $C, g$

$$R^2_{true} = \frac{G^2}{g^T C^{-1} g}$$

## 20.1.4 Predicted & Realized Risk

Naive: Sub $E$ for $C$

$$\Rightarrow \pi_E = G \frac{E^{-1} g}{g^T E^{-1} g}$$

$$R^2_{in} = \frac{G^2}{g^T E^{-1} g}$$ "in-sample risk"

$g^T E^{-1} g$ is convex w.r.t. $E$

$$\Rightarrow \mathbb{E}[g^T E^{-1} g] \geq g^T \mathbb{E}[E]^{-1} g = g^T C^{-1} g$$

$$\Rightarrow R^2_{in} \leq R^2_{true}$$

For $E'$ out of sample

$$R^2_{out} := \pi_E^T E' \pi_E^T = G^2 \frac{g^T E^{-1} E' E^{-1} g}{(g^T E^{-1} g)^2}$$

$$\pi_E \perp E' \Rightarrow \text{as } N \to \infty \quad \pi_E^T E' \pi_E = \pi_E^T C \pi_E$$

From optimality of $\pi_C$: $\pi_C C \pi_C \leq \pi_E C \pi_E = \pi_E E' \pi_E$

$$\Rightarrow \boxed{R^2_{in} \leq R^2_{true} \leq R^2_{out}}$$

## 20.2 The High-Dimensional Limit

### 20.2.1 $R^2_{in}$ vs $R^2_{out}$ : Exact Results

Let $C$, $gg^T$ free (not a natural assumption unless predictors are market neutral & idiosyncratic characteristics)

$M$ pos definite $\Rightarrow$ $\frac{g^T M g}{N} = \frac{1}{N} \mathrm{Tr}[M g g^T] = \frac{g^2}{N} \tau(M)$

Can normalize $g^2/N = 1$

1) $R^2_{in} = \frac{G^2}{N \tau(E^{-1})}$

2) $R^2_{true} = \frac{G^2}{N \tau(C)}$

3) $R^2_{out} = \frac{G^2 \tau(E^{-1} C E^{-1})}{N \tau(E^{-1})^2}$

1) & 2): $q < 1$

$\tau(C^{-1}) = (1-q)\,\tau(E^{-1}) \Rightarrow R^2_{in} = \underbrace{(1-q)}_{\text{precise quantification of over-optimism}} R^2_{true}$ ← 0 as $q \to 1$

3): $E = C^{1/2} W_q C^{1/2}$

$\Rightarrow R^2_{out} = \frac{G^2 \tau(C^{-1} W_q^{-2})}{N \tau(E^{-1})^2}$

W, C asymptotically free $= \frac{G^2}{N}(1-q)^2 \frac{\tau(C^{-1})\, \overbrace{\tau(W_q^{-2})}^{(1-q)^{-3}}}{\tau(C^{-1})^2}$

$= \frac{R^2_{true}}{1-q}$ ← diverges as $q \to 1$

$\frac{R^2_{in}}{1-q} < R^2_{true} < (1-q) R^2_{out}$

$q \to 1$ is most dangerous

## 20.2.2 Out-of sample Risk Minimization

Naively since Markowitz uses $C^{-1}$, may think to estimate that

but $\mathbb{E}\, R^2_{out}$ depends linearly on $C \Rightarrow$ estimate that

Proof:

$$\Xi = \sum_j \xi(\lambda_j)\, \underset{\text{evecs of } E}{v_j} v_j^T$$

$$R^2_{out} = G^2 \frac{Tr(\Xi^{-1} C \Xi^{-1})}{(Tr\, \Xi^{-1})^2} = G^2 \sum_j \frac{v_j^T C v_j}{\xi^2(\lambda_j)} \left(\sum_j \frac{1}{\xi(\lambda_j)}\right)^{-2}$$

$$\frac{\partial R^2_{out}}{\partial \xi_j} = 0 \Rightarrow -2\frac{v_j^T C v_j}{\xi(\lambda_j)^3}\left(\sum \frac{1}{\xi}\right)^{-2} + \frac{2}{\xi(\lambda_j)^2} \sum \frac{v^T C v}{\xi^2}\left(\sum \frac{1}{\xi}\right)^{-3}$$

$$\Rightarrow \xi(\lambda_j) = \underset{\text{const}}{A}\; v_j^T C v_j$$

$Tr[\Xi] = Tr[C] \Rightarrow A = 1 \Rightarrow \xi$ is oracle estimator

$\Rightarrow$ RIE minimizes $R^2_{out}$!

$$Tr[\Xi^n C] = \sum_j \xi(\lambda_j)^n\, Tr[v_i v_i^T C] = \sum_j \xi^n(\lambda_j)\, \underbrace{v_i^T C v_i}_{\xi(\lambda_j)} = Tr[\Xi^{n+1}]$$

$\forall n \in \mathbb{Z}$

$$\Rightarrow \quad R^2_{out}(\Xi) = \frac{G^2}{Tr(\Xi^{-1})}$$

## 20.2.3 Inverse Wishart Model

$C = M_p \qquad p > 0$

$\tau(C^{-1}) = -g_C(0) = 1 + p$

$$\Rightarrow R^2_{true} = \frac{G^2}{N} \frac{1}{1+p} \quad \text{as } N \to \infty$$

Can show $\tau(\Xi^{-1}) = -(1+\frac{q}{p})\, g_E(-\frac{q}{p}) = 1 + \dfrac{p^2}{p+q+pq}$

$$\Rightarrow \mathcal{R}^2_{out}(\Xi) = \frac{\mathcal{G}^2}{N} \frac{p+q+pq}{(p+q)(1+p)}$$

$$\Rightarrow \frac{\mathcal{R}^2_{out}(\Xi)}{\mathcal{R}^2_{true}(\Xi)} = 1 + q \frac{pq}{(p+q)(1+p)} \geq 1$$

$$\mathcal{R}^2_{in}(\Xi) = \frac{\mathcal{G}^2}{N} \frac{\tau(\Xi^{-1} E \Xi^{-1})}{\tau(\Xi^{-1})^2}$$ Bun et al 2017

$$= \frac{\mathcal{G}^2}{N} \frac{p+q}{(1+p)(p+q(p+1))}$$

$$\Rightarrow \frac{\mathcal{R}^2_{in}(\Xi)}{\mathcal{R}^2_{true}(\Xi)} = 1 - \frac{pq}{p+q(1+p)} \leq 1$$

Can see $\mathcal{R}^2_{in}(\Xi) - \mathcal{R}^2_{in}(E) \geq 0$

$\mathcal{R}^2_{out}(\Xi) - \mathcal{R}^2_{out}(E) \leq 0$



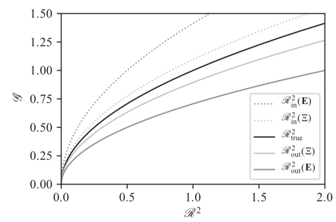



Figure 20.1 Efficient frontier associated with the mean-variance optimal portfolio (20.10) for **g** = **1** and **C** an inverse-Wishart matrix with $p = 0.5$, for $q = 0.5$. The black line depicts the expected gain as a function of the *true* optimal risk (20.11). The gray lines correspond to the realized (out-of-sample) risk using either the SCM **E** or its RIE version $\Xi$. Both estimates are above the true risk, but less so for RIE. Finally, the dashed lines represent the predicted (in-sample) risk, again using either the SCM **E** or its RIE version $\Xi$. $\mathcal{R}$ and $\mathcal{G}$ in arbitrary units, such that $\mathcal{R}_{true} = 1$ for $\mathcal{G} = 1$.

## 20.3 Statistics of Price Changes

Bachelier's thesis: price variations $\propto \sqrt{\text{time}}$

$$V(\tau) := \mathbb{E}\left[\left(\log \frac{p_{t+\tau}}{p_t}\right)^2\right]$$

$$\Rightarrow V(\tau) = \sigma^2 \tau$$

Now let $\log p_t = \log p_0 + \sum_{t'=1}^{t} r_{t'}$

$$\mathrm{Cov}(r_{t'}, r_{t''}) = \sigma^2 C_r(|t' - t''|)$$

$C_r(u) = \delta(u)$ for uncorrelated walk

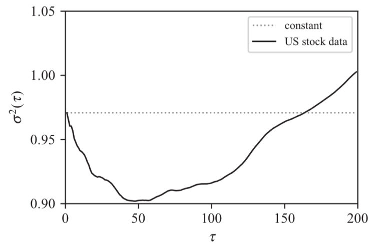
Trending $\Rightarrow C_r(u) > 0$ Mean-reverting $\Rightarrow C_r(u) < 0$

## Implications for Bachelier's first law:

*

$$\sigma^2(\tau) = \frac{V(\tau)}{\tau} = \sigma^2(1)\left[1 + \sum_{u=1}^{\tau}\left(1-\frac{u}{\tau}\right)C_r(u)\right]$$

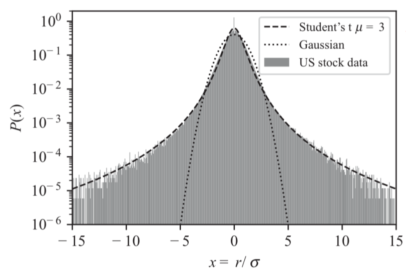





remarkably flat

Figure 20.2 Average signature plot for the normalized returns of US stocks, where the x-axis is in days. The data consists of the returns of 1725 US companies over the period 2012–2019 (2000 business days), returns are normalized by a one-year exponential estimate of their past volatility. To a first approximation $\sigma^2(\tau)$ is independent of $\tau$. The signature plot allows us to see deviations from this pure random walk behavior. One can see that stocks tend to mean-revert slightly at short times ($\tau < 50$ days) and trend at longer times. The effect is stronger on the many low liquidity stocks included in this dataset.

## Returns have fat power law tails:




$\mu = 3$ consistently across markets

Cool!

Figure 20.3 Empirical distribution of normalized daily stock returns compared with a Gaussian and a Student's t-distribution with $\mu = 3$ and the same variance. Same data as in Figure 20.2.

However, returns are far from IID draws from Student's t

Uncorrelated but dependent

Because time-aggregated returns do not revert to a Gaussian

## Volatility is itself time-varying ⇒ heteroskedastic returns

$r_t = \sigma_t \varepsilon_t$ $\quad$ $\varepsilon_t$ are IID non-gaussian of $\mathrm{Var}\,\varepsilon_t = 1$

not indep ⇒ Leverage Effect

For $\mathbb{E}[\sigma_t^2 \sigma_{t+\tau}^2]$

$\mathbb{E}[\varepsilon_t \sigma_{t+\tau}] < 0$

$\mathbb{E}[\varepsilon_t \sigma_{t-\tau}] \approx 0$

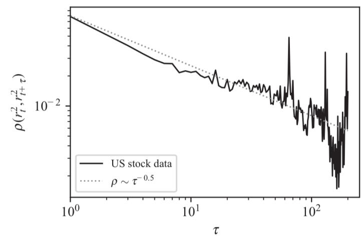



Figure 20.4 Average autocorrelation function of squared daily returns for the US stock data described in Figure 20.3. The autocorrelation decays very slowly with the time difference $\tau$. A power law $\tau^{-\gamma}$ with $\gamma = 0.5$ is plotted to guide the eye. Note the three peaks at $\tau = 65$, 130 and 195 business days correspond to the periodicity of highly volatile earning announcements.

$\mathbb{E}[\sigma_t^2 \sigma_{t+\tau}^2]$

## 20.4 Empirical Covariance

Choose $N = 500$ stocks over $T = 2000$ days

$\Rightarrow q = 1/4$

$\lambda_1 := \lambda_{max} \sim 100 \times \langle \lambda \rangle$

outlier "market mode"

$v_1 \approx 1/\sqrt{N}$ $\quad \left(v_1 \cdot \frac{1}{\sqrt{N}}\right) \approx 0.95$

One-factor model:

$r_{i,t} = \beta_i f_t + \varepsilon_{i,t}$ $\qquad f_t,\ \varepsilon_{it}$ uncorrelated mean 0

$\Rightarrow C_{ij} = \beta_i \beta_j \sigma_f^2 + \delta_{ij} \sigma_\varepsilon^2$ $\qquad \beta \approx \frac{1}{\sqrt{N}}$

We know the spectrum is

1) MP "sea" between $(1 \pm \sqrt{q})^2 \sigma_\varepsilon^2$

2) Outlier at $\sigma_\varepsilon^2 (1+a)(1+\frac{q}{a})$ $\qquad a = \frac{\sigma_f^2 |\beta|^2}{\sigma_\varepsilon^2}$

$|\beta|^2 \sim O(N)$ for large portfolios $\Rightarrow \propto \sigma_\varepsilon^2 q = \sigma_f^2 |\beta|^2$

For $q = 1/4$ get $\lambda_- = 0.2$ $\quad \lambda_+ = 1.8$

We get $\approx 20$ outliers $\Rightarrow$ need more factors $\S$ market sectors

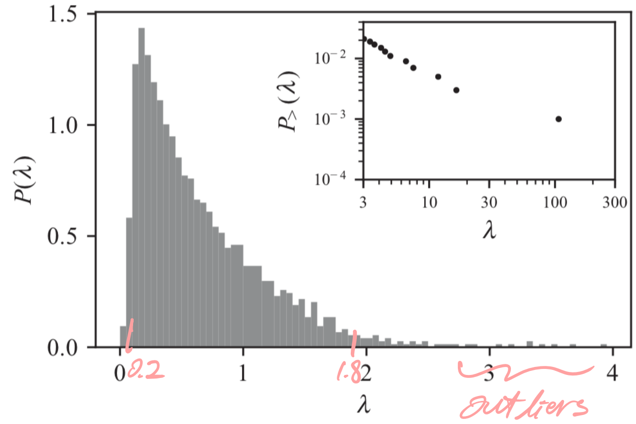



Figure 20.5 Eigenvalue distribution of the SCM, averaged over for three random sets of 500 US stocks, each measured on 2000 business days. Returns are normalized as in Figure 20.3, corresponding to $\bar{\lambda} = 0.97$. The inset shows the complementary cumulative distribution for the largest eigenvalues indicating a power-law behavior for large $\lambda$, as $P_>(\lambda) \approx \lambda^{-4/3}$. Note the largest eigenvalue $\lambda_1 \approx 0.2N$, which corresponds to the "market mode", i.e. the risk factor where all stocks move in the same direction.

becomes linear again for large $\lambda$ outside $\xi$ supp(P), consistent w/ theory.



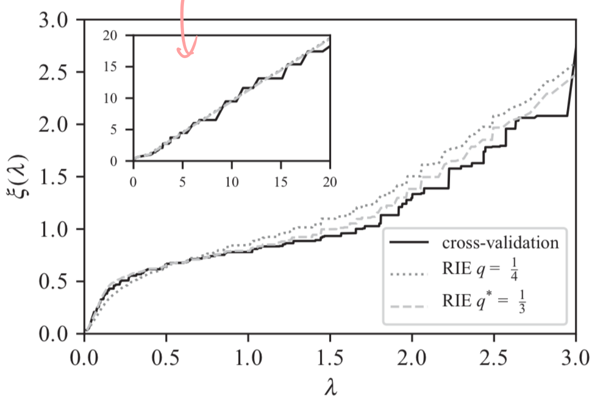



Figure 20.6 Non-linear shrinkage function $\xi(\lambda)$ computed using cross-validation and RIE averaged over three datasets. Each dataset consists of 500 US stocks measured over 2000 business days. Cross-validation is computed by removing a block of 100 days (20 times) to compute the out-of-sample variance of each eigenvector (see Eq. (19.88)). RIE is computed using the sample Stieltjes transform evaluated with an imaginary part $\eta = N^{-1/2}$. Results are shown for $q = N/T = 1/4$ and also for $q^* = 1/3$, chosen to mimic the effects of temporal correlations and fluctuating variance that lead to an effective reduction of the size of the sample (cf. Section 17.2.3). All three curves have been regularized through an isotonic fit, i.e. a fit that respects the monotonicity of the function.